Geometry–Do, White Belt Chapter

Victor Aguilar
United States of America, orcid.org/0009-0007-4646-9890

Abstract: *Geometry–Do* is a textbook about plane geometry. It will be divided into two volumes, *Geometry without Multiplication: White through Red Belt*, and *Geometry with Multiplication: Blue through Black Belt*. The white- and yellow-belt chapters are neutral geometry; the remainder of *Volume One* and all of *Volume Two* is Euclidean geometry. It is primarily intended to teach geometry from the ground up, starting with the postulates and citing only already-proven theorems. It trains mathletes for competition, but it is not the usual grab-bag of unproven theorems chosen haphazardly and solely because they appeared in past exams. The early chapters prepare students for jobs in construction, architecture, surveying, graphic arts, and military defense. The later chapters teach geometry needed by engineers and military officers. In this lecture, the White Belt chapter is presented. I will address these people:

Pure Mathematicians	Moise derides the "lighthearted use of the word *let*." I prove the crossbar theorem and other foundations not usually taught in high school, and I discuss Hilbert's *Foundations of Geometry*.
High-School Teachers	Randomly assigning letters to points is what makes geometry confusing. I have special symbols for midpoints, perpendicular feet, and infeet (where the angle bisector cuts the opposite side of a triangle) and exfeet.
Administrators	I present clear distinctions between *Geometry–Do* and *Common Core* with examples that concerned parents can understand.
Construction Workers	I invent the Aguilar A-Frame, give detailed instructions on squaring a basement foundation wider than a tape measure without exiting the rectangle, and discuss how building with wood differs from steel construction.
Military Officers	I discuss troop positioning along a frontier that is plagued with cross-border raids, which assumes that friendly and enemy troops move at the same speed, and a parabola is the set of points equidistant from the focus and the directrix.

Keywords: Geometry Applications, Common Core Math, Neutral Geometry, STEM, David Hilbert

Citation: Aguilar, V. (2023). Geometry–Do, White Belt Chapter. In M. Shelley, O. T. Ozturk, & M. L. Ciddi, *Proceedings of ICEMST 2023-- International Conference on Education in Mathematics, Science and Technology* (pp. 157-204), Cappadocia, Turkiye. ISTES Organization.

Introduction

This article is divided into two parts. The first part is the introduction to *Geometry–Do*, which discusses the foundations of geometry. The second part is the White Belt chapter, which proves some basic theorems. Every chapter, including White Belt, has appendices that discuss applications. The main text of each chapter strictly follows the axiomatic method with each theorem being proven citing only the postulates and previously proven theorems. But the chapter appendices sometimes call on information that students are expected to know from their previous studies; *e.g.* the appendix about defense positioning assumes prior knowledge of the equation for a parabola. Parabolas are taught in Algebra I, which precedes Geometry in most high schools.

The main text of *Geometry–Do* does not have any prerequisites – we are doing *Volume One* without multiplication – though the extended chains of reasoning will be best met by students with some prior study of mathematics. But White Belt is basic, so any high-school student should be able to read this article. If readers wish to continue, the rest is here:
www.researchgate.net/publication/291333791_Volume_One_Geometry_without_Multiplication

Euclid's Postulates Plus One More

Segment	Two points fully define the **segment** between them.
Line	By **extending** it, a segment fully defines a **line**.
Triangle	Three **noncollinear** points fully define a **triangle**.
Circle	The center and the **radius** fully define a **circle**.
Right Angle	All **right angles** are equal; equivalently, all **straight angles** are equal.
Parallel	A line and a point not on it fully define the **parallel** through that point.

Segments are denoted with a bar, $\overline{EF}$; **rays** with an arrow, $\overrightarrow{EF}$, which have endpoint E and are extended on the F side infinitely; lines with a double arrow, $\overleftrightarrow{EF}$, which are extended infinitely both ways; and **angles** as $\angle EFG$ or $\angle F$ if there is only one angle at F. Triangles and quadrilaterals are also denoted with bars, as $\overline{EFG}$ and $\overline{EFGH}$. The **postulates** are in terms of **fully defined**, which means that a **figure** with the given characteristics exists, and it is unique. **Under defined** means figures with the given characteristics are legion. John Playfair stated the parallel postulate as I and David Hilbert do, which is **equivalent** to Euclid's Fifth Postulate (Euclid, 2013, p. 2).

> *If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

While Hilbert and I both found Euclid's postulate to be convoluted and chose Playfair's version, and we both reject real numbers as unsupported by our postulates, we otherwise are different.

Euclid also had five "common notions," which vaguely describe what modern mathematicians call equivalence relations, total orderings, and additive groups.

Equivalence Relations and Total Orderings

A **relation** is an operator, $\mathcal{R}$, that returns either a "true" or a "false" when applied to an ordered pair of elements from a nonempty set. (We only use binary relations, so we can omit "binary.") Relations must be applied to objects from the same set. For instance, $\overline{EF} = \angle G$ is neither true nor false; it is incoherent. There are four ways that relations may be characterized. For one to hold, it must apply to all possible choices x, y, z from the given set, not just some of them.

Reflexive	$x \mathcal{R} x$
Symmetric	$x \mathcal{R} y$ implies $y \mathcal{R} x$
Anti-Symmetric	$x \mathcal{R} y$ and $y \mathcal{R} x$ implies $x = y$
Transitive	$x \mathcal{R} y$ and $y \mathcal{R} z$ implies $x \mathcal{R} z$

A reflexive, symmetric, and transitive relation is called an **equivalence relation**. The principal equivalence relations considered in geometry are **equality**, $=$, which applies to segments, angles, or areas; **congruence**, $\cong$, which applies to triangles; **similarity**, $\sim$, which applies to triangles; and **parallelism**, $\parallel$, which applies to lines. $\overleftrightarrow{EF} \parallel \overleftrightarrow{GH}$ means that $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$ do not intersect. There are an infinity of points in the plane; strange and useless results can be made of small finite sets.

Since segments are known only by their **length**, $\overline{EF} = \overline{GH}$ means that $\overline{EF}$ and $\overline{GH}$ are the same length. Since length is the same regardless of direction, it is always true that $\overline{EF} = \overline{FE}$. But triangles are known, not by just one magnitude, but by six. The vertices are ordered to show which ones are equal. $\overline{EFG} \cong \overline{JKL}$ implies $\overline{EF} = \overline{JK}$, $\overline{FG} = \overline{KL}$, $\overline{GE} = \overline{LJ}$, $\angle E = \angle J$, $\angle F = \angle K$ and $\angle G = \angle L$; and these equalities imply congruence. Beware! Writing the vertices of a triangle out of order is one of the most common mistakes made by beginning geometers.

A **quadrilateral** is a union of two triangles adjacent on a side such that it is **convex**; congruence or similarity holds if and only if both pairs of triangles are congruent or similar. If $\overline{EFG} \cong \overline{JKL}$ and $\overline{EHG} \cong \overline{JML}$, then, $\overline{EFGH} \cong \overline{JKLM}$. Analogously, if $\overline{EFG} \sim \overline{JKL}$ and $\overline{EHG} \sim \overline{JML}$, then, $\overline{EFGH} \sim \overline{JKLM}$. Similarity is defined as two triangles with all corresponding angles equal, so $\overline{EFG} \sim \overline{JKL}$ and $\overline{EHG} \sim \overline{JML}$ means that six pairs of corresponding angles are equal. This is more than just saying that the four corresponding **interior** angles of $\overline{EFGH}$ and $\overline{JKLM}$ are equal; thus, it is not true that proving these four angles equal is sufficient to prove $\overline{EFGH} \sim \overline{JKLM}$. A counter-example is a right square and rectangle; they have all right angles, but they are not similar. "Four-sided figure" is a vacuous quadrilateral definition that leads beginners to err by claiming that right squares and rectangles are similar. We make quadrilaterals a logical extension of triangles.

Relations that are anti-symmetric can only be defined if we have already defined equality, because equality is referenced in its definition. (Equality is the only relation that is both symmetric and anti-symmetric.) A relation that is not symmetric but has the other three characteristics is called a **total ordering**. The adjective total is redundant because we said relations must hold for every pair of elements. (Partial orderings, such as subset, exist in other branches of mathematics.) Geometers only use less than or equal to, $\leq$. ($\geq$ could be, though we usually order from small to large; $<$ and $>$ are irreflexive and so are not orderings.) A nonempty set with both an equivalence relation, $=$, and a total ordering, $\leq$, is called a **magnitude**. Geometers consider three magnitudes: lengths, angles, and areas.

Note that our definition of magnitude does not imply that real numbers can be associated with lengths, angles, or areas; only that the relations $=$ and $\leq$ exist and have the required properties. (In real-life applications I use integer lengths, denoted by absolute value, *e.g.*, $|\overline{EF}| = 5$ m.) It does imply that magnitudes are unique, which is what the replication **axiom** below is stating.

Equal magnitudes are an equivalence relation and can be reproduced wherever needed; that is, compasses do not collapse when lifted from the paper but are like holding a chain at a length. Compasses that collapse would be like surveyors who can walk a chain around an **arc** but, the moment the center guy moves, their chain turns to smoke. This is a parlor game, not a science!

An **equivalence class** is defined as a subset of all the elements that have an equivalence relation with each other. It can be shown that any two equivalence classes either coincide or are disjoint, hence the collection of equivalence classes form a partition of the set. For example, if the set is all the lines in the plane, it is partitioned by parallelism; each equivalence class is composed of lines stacked on top of each other (parallel) but tilted relative to the lines in the other classes. Equivalence classes can be defined in reference to an existing equivalence class. For instance, if an equivalence class is defined as all the angles equal to a given angle, then all the angles **complementary** to any member of that class are equal to each other; that is, they form their own

equivalence class. All the angles **supplementary** to any member of that class are also equal to each other. If an equivalence class is defined as all the lines parallel to a given line, then all the lines **perpendicular** to any member of that class are parallel to each other. All the circles with radii equal to any member of an equivalence class of equal segments are an equivalence class.

Equivalence also refers to statements that can be proven if the other one is assumed, and in either order. For instance, Euclid's fifth postulate and Playfair's postulate are equivalent because, assuming either to be true, it is possible to prove that the other is true. The equivalence of **theorems** can be expressed by separating them with the phrase "if and only if," which can be abbreviated "iff." Proof in the other direction is called the **converse**; that is, if p implies q, then the converse is that q implies p. If p and q are equivalent, then both implications are true.

Proof by contradiction when there is only one alternative that must be proven impossible is called a **dichotomy**. A **trichotomy** (*e.g.* ASA congruence) has three alternatives. A magnitude can either be less than, equal to or greater than another, and only one of these three is desired; thus, by proving the other two to be impossible, we know that it is the one that makes the theorem true.

Additive Groups

We define an additive group as a nonempty set that is closed under an operation that we will denote $+$ and which has these properties for all x, y, z that are members of that set:

Associative property	$(x+y)+z=x+(y+z)$
Commutative property	$x+y=y+x$
Existence and uniqueness of an identity	$x+0=x=0+x$
Existence of unique inverses (identity is its own)	$x+(-x)= \ 0 \ =(-x)+x$

There exist magnitudes that are not additive groups, such as economic value. Given a choice between x or y, it is always possible for a person to choose one. But, because x may substitute for or be a complement to y, they are not independent the way geometric magnitudes are. There are also additive groups that cannot be ordered, such as matrices. Matrices of the same dimension are an additive group, but we cannot say $\boldsymbol{X} \leq \boldsymbol{Y}$ for any two distinct matrices.

On the first day of class I ask the students to look back to a time eight or ten years prior, when they were little kids and knew only how to add and subtract; multiplication and division was still scary for them. I assure them that geometry will be like going back to 1st grade. Sticking segments together end to end or angles together side by side is no more difficult than 1st grade problems about adding chocolates to or subtracting chocolates from a bowl of candies. How easy is that?

Replication Axiom

Given $\overline{EF}$ and $\overrightarrow{JK}$, there exists a unique point L on $\overrightarrow{JK}$ such that $\overline{EF} = \overline{JL}$.

Given $\angle EFG$ and $\overrightarrow{KJ}$, there exist rays $\overrightarrow{KL}$ and $\overrightarrow{KL''}$ such that $\angle EFG = \angle JKL = \angle JKL''$.

The symbol $<$ is defined by the terms "between" and "inside," as stated in the two axioms below. But this symbol can also be applied to magnitudes. $|\overline{EF}| < |\overline{JK}|$ means that the number of units that can be laid off inside $\overline{EF}$ is less than the number that can be laid off inside $\overline{JK}$. The absolute value signs denote these numbers, so we use $+$ when combining them; $|\overline{EF}| + |\overline{JK}|$ is the sum of these lengths. Degrees or radians measure angles and can be added, but that is undefined in this book; it is trigonometry. We measure area though; $|\overline{EFG}| + |\overline{JKL}|$ is their combined area.

Interior Segment Axiom

*If M is **between** E and F, then $\overline{EM} < \overline{EF}$ and $\overline{MF} < \overline{EF}$ and $\overline{EM} \cup \overline{MF} = \overline{EF}$.* ($\cup$ *means union.*)

Interior Angle Axiom

*If P is **inside** $\angle EFG$, then $\angle EFP < \angle EFG$ and $\angle PFG < \angle EFG$ and $\angle EFP \cup \angle PFG = \angle EFG$.*

Axioms and Foundational Theorems

For a point to be between E and F means to be on the segment they define, $\overline{EF}$, but at neither **endpoint**. To be inside $\angle EFG$ (not straight) means to be between points on $\overrightarrow{FE}$ and on $\overrightarrow{FG}$, with neither point being F. It is instinctive that all humans know what it means for a point to be between two points and – in the case of Pasch's axiom – also what it means for a segment to be continuous; that is, with no gaps where another segment might slip through. Triangles and quadrilaterals are defined to be convex; this means that they are not allowed to be concave or **degenerate**. Interior angles are greater than zero and less than straight (indeed, all angles are because of "between" in the definition), so triangles are never segments, and quadrilaterals are never triangles or darts.

Pasch's Axiom

If a line passes between two vertices of a triangle and does not go through the other vertex, then it passes between it and one of the two vertices.

In *Geometry–Do*, plane, point, shortest path and straight are **undefined terms**. These are concepts that a parent does not have to explain to a child; they are just giving names to what is already in the child's mind. Specifically, a plane is undefined because rigorously defining uncountably infinite, flat, and of exactly two dimensions is beyond the scope of this book. Euclidean **area** is defined as the measure of the size of a triangle

or a union of **disjoint** triangles. Like the ancients, we do not have a rigorous definition of limits but just rely on intuition; wheat plants are infinitesimal compared to fields, so weighing the wheat is almost like calculating a limit. Thus, area too is something that small children can understand without explanation. Defining area as the product of a right rectangle's sides waits for *Volume Two: Geometry with Multiplication*. This definition of area is not intuitive to small children, who know nothing of multiplication. For now, just know that area is a magnitude.

Degrees of angle or radians will not be defined in either volume because doing so is trigonometry.

Triangle Inequality Theorem ***(Euclid, Book I, Prop. 20, 22)***
Three lengths can be of triangle sides if and only if the sum of the lengths of any two sides is greater than the length of the third side.

In ancient Greece, Epicurus scoffed at Euclid for proving a theorem that is evident even to an ass (donkey), who knows what the shortest path to a pile of hay is. Some textbooks call it an axiom, and some prove only one direction – they start with the existence of the triangle and prove the inequalities – but that is not the direction needed for SSS, which cites it. Beginners here should just take it as an axiom; also, they should take the continuity theorem (below) as an axiom. Its proof requires the Cantor axiom, which assumes a knowledge of set theory that is not expected of beginning geometers. Experts can find detailed proofs in an appendix at the end of this book.

Continuity Theorem

1. *A line that passes through a point inside a circle intersects the circle exactly twice.*
2. *A circle that passes through points inside and outside a circle intersects it exactly twice.*

The foundations explained above are sufficient through red-belt study. In these early chapters, students will learn to bisect, trisect and quadrisect a segment, and to multiply it by small natural numbers by using repeated addition. No more of these repeated additions are needed than four, for construction of the Egyptian or $3 : 4 : 5$ right triangle, except that we mention in passing the $5 : 12 : 13$ right triangle, which is used by plumbers when installing $22.5°$ elbows. Elementary school teachers are wrong when they define multiplication as repeated addition; this is why so many students are later confounded by real numbers like $\sqrt{2}$ or π. The repeated addition used in $3 : 4 : 5$ right triangles has nothing to do with multiplying lengths as defined in *Volume Two*.

Blue belts will learn of similarity and prove the triangle similarity theorem. They will go beyond bisecting and trisecting segments to constructing segments whose length relative to a given unit is any rational number. Another axiom is needed for this. A nonempty set with both an equivalence relation, $=$, and a total ordering, $\leq$, is called a magnitude. But to construct segments whose length relative to a given unit is any rational number, length must also be Archimedean.

Archimedes' Axiom

Given any two segments $\overline{EF} < \overline{GH}$, there exists a natural number, n, such that $n|\overline{EF}| > |\overline{GH}|$.

This may seem trivially true, but Galois (finite) fields are not Archimedean. Every schoolboy is taught that Archimedes claimed that, given a long enough lever and a fulcrum to rest it on, he could move the world. They typically receive no clear answer from their teacher on why it matters, since no such fulcrum exists, and Archimedes seems to ignore that gravity is attractive. The point that Archimedes is making is that, if there were such a fulcrum and much gravity under it, he would need a lever 6×10^{22} longer on his side of the fulcrum to balance his mass against the Earth. If the fulcrum were one meter from Earth, Archimedes would be in the Andromeda galaxy if he stood on the other end of that long lever. 6×10^{22} is a big number, but it does exist.

We said above that undefined terms are concepts that one does not have to explain to a child; the adult is just giving names to concepts that are already in the child's mind. But defining natural numbers as $1, 2, 3, \ldots$ is only intuitive up to as many fingers as the child has. We think 6×10^{22} exists because countably infinite fields are consistent; but so are big Galois fields. This axiom is why it is traditional in America to tell children that every snowflake is unique; it helps them visualize big numbers. (Dinosaurs help them visualize vast gulfs of time.) That Archimedes' axiom is not intuitive to small children is one reason why similarity is delayed until blue belt. Note that visualizing big numbers – the vast number of snowflakes in just one field – is what this parable meant when I was a child. But lately, psychologists have commandeered this expression in their happy talk for depressed people, resulting in the sneering retort, "Well, aren't *you* a special snowflake!" So, be careful when mentioning this parable in the context of the Archimedes axiom!

But these are issues of concern to black belts; first, the student must take a short jog through the colored belts, which are concerned with what Mihalescu (2016) refers to as the remarkable elements of triangles and quadrilaterals. By this we initially mean the principal triangle centers. The medians intersect at the medial point, the angle bisectors intersect at the incenter, the altitudes intersect at the orthocenter, and the mediators intersect at the circumcenter. In this introduction the student does not need to know what any of these things are, only that medians – the segment from a vertex to the midpoint of the **opposite** side – and the bisectors of vertex angles are always inside their vertex angle. In $\overline{EFG}$, if the vertex is F, then they are inside $\angle EFG$.

Crossbar Theorem

Given triangle $\overline{EFG}$ and a point P inside it, the ray $\overrightarrow{EP}$ intersects the segment $\overline{FG}$.

> ***Proof***
>
> Let Q be on the ray $\overrightarrow{GE}$ so E is between G and Q. Consider the triangle $\overline{QFG}$. The line $\overleftrightarrow{EP}$ passes between the vertices Q and G because E is between G and Q, and it does not pass through the other

vertex, F, because P is inside $\overline{EFG}$, which means that it is not on $\overline{EF}$. Thus, the conditions of Pasch's axiom are met and $\overleftrightarrow{EP}$ must intersect either $\overline{FG}$ or $\overline{FQ}$. By construction, $\overrightarrow{EP}$ and $\overline{FQ}$ are on **opposite** sides of $\overleftrightarrow{EF}$, so $\overrightarrow{EP}$ cannot intersect $\overline{FQ}$. The ray in the other direction of $\overrightarrow{EP}$ does not intersect either $\overline{FG}$ or $\overline{EF}$ because both segments are on the other side of $\overleftrightarrow{GQ}$. Thus, $\overrightarrow{EP}$ intersects $\overline{FG}$.

■

The midpoints of segments are denoted by the letter M with a double subscript, which are the endpoints of the segment. Thus, two medians of the triangle $\overline{EFG}$ are $\overline{EM_{FG}}$ and $\overline{FM_{GE}}$. Consider the triangle $\overline{EM_{FG}G}$. The line $\overleftrightarrow{FM_{GE}}$ passes between the vertices G and E because M_{GE} is between G and E, and it does not pass through the other vertex, M_{FG}, because M_{FG} is not F. Thus, the conditions of Pasch's axiom are met and $\overleftrightarrow{FM_{GE}}$ must intersect either $\overline{EM_{FG}}$ or $\overline{M_{FG}G}$. Since it intersects $\overleftrightarrow{M_{FG}G}$ at F, it cannot also intersect this line in the segment $\overline{M_{FG}G}$. Thus, it intersects $\overline{EM_{FG}}$. This proves that the medial point of a triangle is always inside the triangle. Note that midpoints are defined in C. 1.2, and their existence inside the segment assured. Triangle centers will be defined later, and their existence assured. On this page we speak casually of things that will be treated rigorously later.

Analogously, the incenter of a triangle is always inside the triangle. The only difference in the proof is that, instead of knowing that the bisectors of vertex angles E and F intersect the opposite sides at M_{FG} and M_{GE}, respectively, we must first invoke the crossbar theorem to prove that they intersect the opposite sides *somewhere* on them, and give these points labels; say, E^* and F^*.

By the triangle postulate, three noncollinear points fully define a triangle and, since the medial point and the incenter have now been proven to be inside the triangle, they are fully defined. Because we nowhere invoked the parallel postulate, medial points and incenters always exist in **neutral geometry** and are thus topics of discussion for white and yellow belts. But what about the orthocenter? A triangle's apex altitude is inside it only if the base angles are **acute**, so white and yellow belts may only discuss the orthocenter if the triangle is known to be acute. By a somewhat more involved argument, the circumcenter also exists for acute triangles. Sometimes these centers exist for triangles that are slightly **obtuse**, though giving a precise meaning to "slightly obtuse" is beyond the scope of this book; thus, white and yellow belts are advised to just defer most discussions of these triangle centers to orange belt.

An Example Theorem with Proof

This concludes our discussion of the postulates of geometry. But you may still be wondering, what is geometry about? The first line of a book is often the only thing people remember about it; for instance, "Call me Ishmael" is the first line of *Moby Dick*. I think it is about a whale – I don't remember. Like Herman Melville, Euclid is

also famous for his first line – but not in a good way. "A point is that which has no part." Beginning geometry students are like, "Oh, so this is a book about Japanese koans?" In the first paragraph of *Geometry–Do*, there are a dozen boldface terms for the student to look up in the glossary, so you may be thinking, "Oh, so this is a book about memorizing vocabulary? It is like learning a language spoken in a country that I will never visit?"

When I was a freshman in college, I rather inadvisably took an upper-division course on groups, rings and fields. Why not? If not with knowledge, I was at least filled with ambition! Most of the material in this introduction came from that textbook, but what I remember most is the first line:

The main business of mathematics is proving theorems.

John Fraleigh (1989) set this sentence between horizontal lines, just as I have done above. He must have thought it important! He is right; the business of mathematicians is proving theorems.

Example Theorem

The sum of quadrilateral ***diagonals*** *exceeds the sum of either pair of* ***opposite*** *sides.*

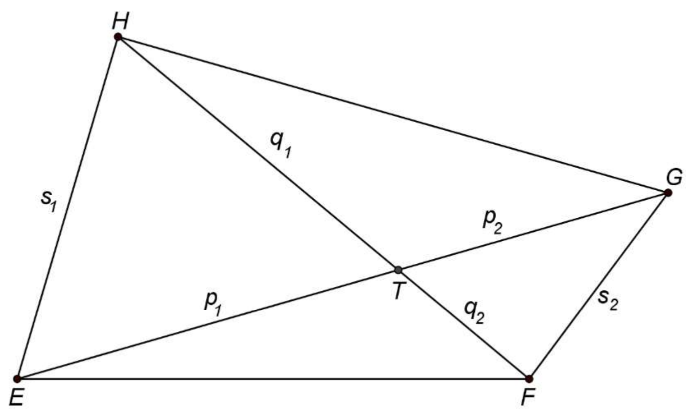
Let us try proving a theorem, and do so now, before the add/drop date, while there is still time for students to make a run for it! This will be fun. We will do it step by step, so I can lead the reader by the hand through a genuine geometry proof. It is easy, and it requires only the basics.

Step One: The first step is to remind yourself of the definitions of terms that you already know, and to look up any terms that you have not yet learned. We read about the quadrilateral earlier; but let us look it up to make sure that we know it. Look up adjacent, convex, and diagonal too!

Quadrilateral	The union of two triangles adjacent on a side such that it is convex; $\overline{EFGH}$
Adjacent	Two disjoint triangles with a common side (common for its full length)
Convex	Any segment between two points interior to two sides is inside the figure
Diagonal	Segments connecting non-consecutive quadrilateral vertices

Step Two: The next step is to draw the figure, and to do so in a way that the definitions of terms are satisfied. For instance, if our two triangles are like the blades on an arrowhead, then the figure is not convex; a segment between the trailing edges of the two blades would not be inside the figure. This is not a quadrilateral; the proof

below does not work for it because its diagonals do not intersect. In the figure below, we see that $\overline{EG}$, the side common to the two triangles, $\overline{EFG}$ and $\overline{GHE}$, is a diagonal; indeed, because it defines the quadrilateral, it is called the **definitional diagonal**. The same quadrilateral can be defined two ways, with two different pairs of adjacent triangles. Sometimes it matters which diagonal is definitional, but the problem at hand mentions both diagonals, so draw both, $\overline{EG}$ and $\overline{FH}$, and label the cut segment lengths p_1, p_2, q_1, q_2. The intersection of the diagonals is labeled T, and two of the side lengths are labeled s_1 and s_2. Lowercase letters denote lengths and can be added; they are not a symbol for the segment itself.



Example Theorem Figure

Step Three. The next step is to go through the index and look for relevant postulates, axioms, and theorems. The index is fifty pages long; so, later, this can be a daunting task. Intuition and experience when carrying out this search is what divides passing green- and red-belt geometers from failing ones. But, at this early stage in your career, the index for the introduction amounts to only two pages, so it is not a long search. The problem is about comparing two sums, so let us remind ourselves about additive groups. By segment addition, the two diagonals are $p_1 + p_2$ and $q_1 + q_2$. Their sum is $(p_1 + p_2) + (q_1 + q_2)$. We are comparing two magnitudes; so, relevant is a theorem about one magnitude being less than another. It is the triangle inequality theorem!

Step Four: The final step is to carry out the proof. There are two triangles with their sides labeled. In $\overline{ETH}$, by the triangle inequality theorem, $s_1 < p_1 + q_1$. In $\overline{FTG}$, by the triangle inequality theorem, $s_2 < p_2 + q_2$. Add the two inequalities together: $s_1 + s_2 < (p_1 + p_2) + (q_1 + q_2)$. We do not have to go through this for the other pair of opposite sides; just say, "analogously."

Thus, you have seen the business of mathematicians. If you have not concluded that being a waiter with a psychology degree is the life for you, then I will see you tomorrow! We will prove the dreaded side–angle–side (SAS) theorem, which stumped both Euclid and David Hilbert!

Experts only!!! (Those uninterested in Hilbert's *Foundations* skip to the notation section.)

Hilbert's "straight line" is redundant; there is no such thing as an unstraight line, so we will just say "line." Hilbert uses the letters $A, B, C, \ldots$ for points, but these will be changed to $E, F, G, \ldots$ to be compatible with *Geometry–Do*, where the first four letters have special meanings. "Always completely determine" is the same thing as "fully define," so we will use the *Geometry–Do* term. "Situated in the same line" means collinear; indeed, "situated on a line" can just be "on a line." "Passes through a point of the segment" means "intersect." I am trying to make this easy!

One reason why beginners are uncomfortable with Hilbert's axioms is their verbosity; sadly, this has often resulted in any mention of foundations being delayed until the students have become advanced. But going back and filling in foundations later is not the right way to teach geometry. This verbosity is largely because German is difficult to translate into English. Economists who have read Carl Menger and philosophers who have read Friedrich Nietzsche have also noticed this. The solution is to not translate quite so literally, which is what I have done below.

I. Axioms of Connection

1. Two distinct points fully define a line.
2. Any two distinct points of a line fully define it.
3. Three points not collinear fully define a plane.
4. Any three points of a plane not collinear fully define the plane.
5. If two points of a line are in a plane, then every point of the line is in the plane.
6. If two planes have a common point, then they have at least one other common point.
7. Lines have at least two points, planes at least three noncollinear points, and space at least four noncoplanar points.

II. Axioms of Order

1. If E, F, G are collinear and F is between E and G, then F is also between G and E.
2. If E and G are two points on a line, then there exists at least one point F that is between E and G and at least one point H so situated that G is between E and H.
3. Of any three collinear points, there is exactly one between the other two.
4. Any four collinear points E, F, G, H can always be arranged so F is between E and G; also, between E and H. Furthermore, G is between E and H; also, between F and H.
5. If E, F, G are not collinear and a line in the plane they determine does not pass through any of them and it intersects $\overline{EF}$, then it will also intersect either $\overline{EG}$ or $\overline{FG}$.

Some jokers have noticed that Euclid never said there had to be more than one point, and so they defined their own geometry with *exactly* one point. Every segment is of zero length and is at the same point; analogously,

every circle has zero radius and has the same center. They took navel gazing to a whole new level, just staring at that one point, and seeing how many geometry theorems are true about it! This must have been meant as a joke, but I think Hilbert was a little too concerned about excluding these degenerate geometry theories and got a bit pedantic doing so with his axioms. *Geometry–Do* is comfortable leaving terms like plane and point undefined and assuming that the students are not going to play any jokes by twisting their meanings; they are just trying to learn some geometry that will be useful in their everyday lives. Thus, I. 7 is not made explicit. II. 1 – 3 are the glossary definition of "between" except that II. 1 has "between" assume collinearity rather than imply it. II. 4 is redundant and is omitted in *Geometry–Do*. II. 5 is Pasch's Axiom, which I include among the secondary axioms at the end of the introduction.

I. 5, 6 are omitted because *Geometry–Do* does not include solid geometry. Two reasons:

1. High-school students have enough on their plates with plane geometry. *Geometry–Do* is a three-year course, assuming I can squeeze blue belt, *Cho–Dan* and *Yi–Dan* into a single year. It is possible, especially if *Sam–Dan* is included, that this will be a four-year course.

2. Traditional solid geometry (*e.g.,* Wentworth or Kiselev) is not very useful. This material is better taught as an application of Calculus III. Wolfe and Phelps have an advanced version of *Practical Shop Mathematics* that is about solid geometry, but it does not cite any of Wentworth's theorems; civilian machinists are just not that into cones and spheres.

Indeed, while Wentworth is harmless, teaching teenagers too much about machining cones and paraboloids is risky because of their use in shaped charges and explosively formed projectiles. In high school, the volume formulas are just food for memorization. In Calculus III, these formulas can be derived, and the students are mature enough not to do anything crazy like making an EFP.

Geometry–Do postulates comparable to Hilbert's *Axioms of Connection and Order*

Segment	Two points fully define the segment between them.
Line	By extending it, a segment fully defines a line.
Triangle	Three noncollinear points fully define a triangle.

We are left with I. 1, 2, 3, 4, which are comparable to my segment, triangle, and line postulates. Hilbert's axioms are about points and lines; he defines segment almost as an afterthought. But *Geometry–Do* follows Euclid by having distinct segment and line postulates. This is wise because segments are foundational; they should not just be tossed in later. Also, I. 1, 2 are redundant; two points define a unique line, and a line is defined by any two points on it is just one postulate. I. 3, 4 is also just one postulate; let us compare it to the triangle postulate of *Geometry–Do*.

Hilbert is overreaching when he states that three noncollinear points fully define a plane. The Euclidean plane and the Lobachevskian plane are different things. Without a parallel postulate, existence of triangle centers is only assured inside the triangle with these vertices. To say, "the plane" requires explanation of what, exactly, has been defined. In the introduction, I write:

> *By the triangle postulate, three noncollinear points fully define a triangle and, since the medial point and the incenter have now been proven to be inside the triangle, they are fully defined. Because we nowhere invoked the parallel postulate in the preceding proofs, medial points and incenters always exist in neutral geometry... But what about the orthocenter? A triangle's apex altitude is inside it only if the base angles are acute, so white and yellow belts may only discuss the orthocenter if the triangle is known to be acute. By a somewhat more involved argument, the circumcenter also exists for acute triangles. Sometimes these centers exist for triangles that are slightly obtuse, though giving a precise meaning to "slightly obtuse" is beyond the scope of this book.*

III Axiom of Parallels

In a plane there can be drawn through any point not on a line, one and only one line that does not intersect the given line. This line is called the line's parallel through that point.

Hilbert's third group of axioms consists of only one axiom, which is the same as in *Geometry.*

IV Axioms of Congruence

1. If E and F are two points on a line and J is a point on the same or another line, then, on a given side of J on this line, there exists a unique point K such that $\overline{EF}$ is congruent to $\overline{JK}$, which is written $\overline{EF} \equiv \overline{JK}$. Every segment is congruent to itself; $\overline{EF} \equiv \overline{EF}$.

I am not an historian; but, as far as I know, this is the first time anyone ever used the term congruent to mean that two segments are the same length. Euclid would have said that they are equal and, as evidenced by Kiselev and Wentworth, this continued to be the practice through the 19th century in both the East and the West. Equal refers to magnitudes because they are fully defined by a single measurement, *e.g.,* the length of a segment. Triangles have three sides and three angles but – only after proving some theorems – we know that it is possible to measure three magnitudes and have equality for all six. Congruence is not just a single measurement.

Also, Hilbert's notation is confusing, though this may be due to the typesetting of his day. He did not use overlines while we use $\overline{EF}, \overrightarrow{EF}$ and $\overleftrightarrow{EF}$ to mean segment, ray, and line, respectively. He used only $\equiv$ while we use $=$, $\cong$ and $\equiv$ to mean equals, congruent and coincident, respectively.

IV Axioms of Congruence

1. Given $\overline{EF}$ and $\overrightarrow{JK}$, there exists a unique point L on $\overrightarrow{JK}$ such that $\overline{EF} = \overline{JL}$.
2. If $\overline{EF} = \overline{JK}$ and $\overline{EF} = \overline{LM}$, then $\overline{JK} = \overline{LM}$.
3. F is between E, G; also, K is between J, L. If $\overline{EF} = \overline{JK}$ and $\overline{FG} = \overline{KL}$, then $\overline{EG} = \overline{JL}$.
4. Given $\angle EFG$ and $\overrightarrow{KJ}$, there exist rays $\overrightarrow{KL}$ and $\overrightarrow{KL''}$ such that $\angle EFG = \angle JKL = \angle JKL''$.
5. If $\angle EFG = \angle JKL$ and $\angle EFG = \angle MNO$, then $\angle JKL = \angle MNO$.
6. If $\angle EFG = \angle JKL$ and $\overline{EF} = \overline{JK}$ and $\overline{FG} = \overline{KL}$, then $\angle FGE = \angle KLJ$ and $\angle GEF = \angle LJK$.

Hilbert's axioms of congruence are here written using *Geometry–Do* notation. I wrote all six of them in six lines, while Hilbert uses a total of 33 lines. This verbosity is one reason why Hilbert is no longer taught to beginners, though this does not justify omitting any discussion of the axiomatic method or just giving it lip service, as is typical these days.

Hilbert is not saying much here. IV. 1, 4 are the replication axiom, IV. 2, 5 are transitivity, and IV. 3 is substitution of equals in addition, which apparently applies only to segments, but not to angles. Frankly, my statement that "a set with both an equivalence relation, $=$, and a total ordering, $\leq$, is called a magnitude" and that there are three geometric magnitudes – lengths, angles, and areas – is a *lot* clearer and more succinct. Also, it is more complete. Why does IV. 3 not have an analogous statement about angles? What about area? Why is only transitivity mentioned and not the reflexive, symmetric and anti-symmetric relations? This is a *very* sketchy description of the properties of equivalence relations, total orderings, and additive groups.

IV. 6 is SAS congruence, though Hilbert makes $\overline{EG} = \overline{JL}$ a theorem. I prove SAS by citing the triangle postulate, which Hilbert could not do because he said that three noncollinear points fully define the plane, which is not what is needed to prove SAS. Hilbert is defining the plane to distinguish it from other planes in the context of solid geometry; we just want to prove SAS.

There is no axiom comparable to my circle postulate; Hilbert just inserts the definition of circle immediately before moving on to Archimedes' axiom, which is a bad idea for the same reason that casually inserting the definition of segment is. Segments and circles are foundational and deserve their own postulates.

Geometry–Do also has Archimedes' axiom; it is among the secondary axioms in the introduction.

Straight angles equal each other if and only if right angles equal each other. Straight is undefined and we could say that it is intuitive that they are all equal, as Hilbert does, or we could use Euclid's postulate, as I do. It is the same thing; Hilbert is too hard on Euclid when he calls him wrong. My right-angle postulate is, "all right angles are equal; equivalently, all straight angles are equal."

Notation

$\alpha,\ \beta,\ \gamma,\ \delta$	Angles of a triangle or quadrilateral; usually $\angle E, \angle F, \angle G, \angle H$, respectively. If α and β are base angles of a triangle, then $\delta = \lvert\alpha - \beta\rvert$, the skew angle.
$\rho,\ \sigma,\ \varphi$	ρ is right, σ is straight, and φ is the interior angle in an equilateral triangle.
$E,\ F,\ G, \ldots, W$	Points. H, I, O, R, S, T, U, V have assigned meanings; do not use arbitrarily.
$M,\ I,\ X,\ Y,\ Z$	M is usually inside a segment; M_{EF} is the midpoint of $\overline{EF}$. Otherwise, double subscripts denote reflection. I is the incenter, X, Y, Z are the excenters and, when subscripted with E, F, G, their pedal points.
$E',\ F',\ G'$	The feet of perpendiculars from E, F, G, particularly the altitudes of $\overline{EFG}$
$E^*,\ F^*,\ G^*$	Infeet; intersections of angle bisectors with the opposite sides of a triangle
$E^\times,\ F^\times,\ G^\times$	Exfeet; intersection of exterior angle bisectors with the opposite sides of a triangle
$e,\ f,\ g$	Lengths of the sides of a triangle opposite the E, F, G vertices, respectively
$a,\ b,\ c$	The coefficients of $ax^2 + bx + c = 0$; use u, v, w for right triangle sides.
$H,\ h,\ h_E,\ h_F,\ h_G$	H is usually a triangle's orthocenter unless it is the fourth vertex of a quadrilateral. h is the height of a triangle or parallelogram if given a base. h_E, h_F, h_G are the altitudes dropped from $E,\ F,\ G$.
$A,\ B,\ C,\ D,\ d$	A is the area of a triangle or quadrilateral, *e.g.*, $\overline{EFG}$ has area $A = \lvert\overline{EFG}\rvert$; B is a solid's base area; C is a triangle's medial point or a parallelogram's bi-medial point; D is the circumdiameter; and d is the indiameter.
P	A point, usually interior. P_E, P_F, P_G are the pedal vertices of P in $\overline{EFG}$.
$L_E,\ L_F,\ L_G$	Long centers of $\overline{EFG}$, where the mediators and angle bisectors meet on ω
$r,\ R$	R is circumradius; r is inradius or other radii if there is no incircle present.

$s,\ S,\ T,\ U,\ V$	s is the semiperimeter; S is the anticenter if T is not; and T is the bi-medial. U and V are the first and second Torricelli points.
$\omega,\ O$	ω (omega) is a circle, usually the circumcircle; O is usually a circle's center
$\equiv,\ \cap,\ \cup,\ -,\ \in, :=$	Coincident, intersection, union, removal, element of a set, assign to a label
$\perp,\ \parallel,\ \nparallel,\ \cong,\ \ncong\ \ \sim$	Perpendicular, parallel, not parallel, congruent, not congruent, and similar
$\lvert P\rvert, \lvert\overline{EF}\rvert, \lvert\overline{EFG}\rvert, \lvert\overline{EFGH}\rvert, \lvert x-y\rvert$	Power of a point, unit length, area, area, absolute value

White Belt Instruction: Foundations

Side–Angle–Side (SAS) Theorem ***(Euclid, Book I, Prop. 4)***

Given two ***sides*** *and the angle θ between them, $0 < \theta < \sigma$, a triangle is fully defined.*

Proof

By the segment postulate, the segments have two endpoints and, since they form an angle $0 < \theta < \sigma$, they share an endpoint. This is three noncollinear points so, by the triangle postulate, the triangle is fully defined. Congruence is transitive, so any two anywhere are congruent.

■

Euclid had five postulates, not six, but proof of his fourth proposition, SAS congruence, relied on superposition, which tacitly assumes a whole slew of additional and unmentioned postulates. Many have cast doubt on Euclid, pointing out that superposition – sliding figures around and flipping them over to position one on top of the other – is nowhere defined.

Robin Hartshorne (2000, p. 2), writes, "Upon closer reading, we find that Euclid does not adhere to the strict axiomatic method as closely as one might hope… The method of superposition… cannot be justified from the axioms… we can develop geometry according to modern standards of rigor." But, when *Common Core* was formulated, Hartshorne was shunted aside because Bill Gates was offering big money to redefine congruence in terms of transpositions – sliding figures around on a computer screen to superimpose them – assuring that geometry ceases to exist the moment a student rises from his school computer. By this definition, is a $3:4:5$ triangle drawn in this book congruent to one drawn on the wall of a 4000-year-old pyramid in Egypt? Neither moved! For that matter, did a figure in this book fly through the air and land on your homework?

Isosceles Triangle Theorem ***(Euclid, Book I, Prop. 5)***

If two sides of a triangle are equal, then their opposite angles are equal.

Proof

Given $\overline{EFG}$ with $\overline{GE} = \overline{GF}$, by SAS, $\overline{FGE} \cong \overline{EGF}$ because $\overline{FG} = \overline{EG}$ and $\angle FGE = \angle EGF$ and $\overline{GE} = \overline{GF}$. By congruence, $\angle EFG = \angle FEG$.

■

Observe that, when we cite SAS, the triangle vertices are ordered by the side, angle and side that are equal; later, in more advanced proofs, we will not write "because" and list the equalities. $\overline{FGE}$ and $\overline{EGF}$ have the same vertices but they are different triangles. $\overline{FGE} \cong \overline{EGF}$ is not a trivial statement proven by reflexivity; it requires proof, and it has important implications. The triangle postulate states that three noncollinear points fully define a triangle, but only in the order given.

Equilateral Triangle Theorem

Given a triangle, the following are equivalent:

1. *It is **equilateral**;*
2. *All interior angles are equal;*
3. *The **medians**, the **altitudes**, and the angle **bisectors** are pairwise coincident;*
4. *The three medians are equal;*
5. *The three altitudes are equal;*
6. *The three angle bisectors are equal.*

Half Equilateral Triangle Theorem

*A triangle is **half equilateral** if and only if it is right and one **leg** is half of the **hypotenuse**.*

Proof of the SSS theorem will use a proof by **contradiction**; that is, show that q not true and p true is contradictory. We have defined dichotomy and trichotomy; now we assume that G and J are distinct and then consider the four places where J can be if it is not G. Like aiming a rifle at a target, there are only five alternatives: a bullseye or a miss to the left, right, above, or below. We show that the latter four are impossible. The **lemma** is based on what "inside" means.

Lemma 1.1

*If a triangle is **inside** another triangle, it has less area.*

Side–Side–Side (SSS) Theorem ***(Euclid, Book I, Prop. 8)***

Given three sides that satisfy the triangle inequality theorem, a triangle is fully defined.

Proof

Given $\overline{EFG}$ and $\overline{EFJ}$ with $\overline{EG} = \overline{EJ}$ and $\overline{FG} = \overline{FJ}$, suppose that G and J are distinct. By lemma 1.1, if J is inside $\overline{EFG}$ or inside the angle **vertical** to $\angle EGF$, then $|\overline{EFJ}| < |\overline{EFG}|$ or $|\overline{EFJ}| > |\overline{EFG}|$, respectively, which implies $\overline{EFG} \ncong \overline{EFJ}$. Suppose J is on the E side of $\overrightarrow{FG}$ but not inside $\overline{EFG}$. $\overline{EG} = \overline{EJ}$, so $\overline{EGJ}$ is isosceles. $\angle EJG = \angle EGJ$ by the isosceles triangle theorem. By analogous reasoning, $\overline{FGJ}$ is isosceles and thus $\angle FGJ = \angle FJG$.

$\angle EJG = \angle FJG + \angle EJF$	and by analogous reasoning	$\angle FGJ = \angle EGJ + \angle FGE$
$\angle EJG > \angle FJG$		$\angle FGJ > \angle EGJ$
$\angle EJG > \angle FGJ$		$\angle FGJ > \angle EJG$

A contradiction; J on the F side of $\overrightarrow{EG}$ but not inside $\overline{EFG}$ is also contradictory.

■

In the following constructions, rays and lines are announced without invoking the line postulate; this is in keeping with our plan to avoid tedious proofs with mincing steps. By construction, midpoints, angle bisectors and perpendiculars to a line through a point are fully defined. Metric geometry textbooks begin with the midpoint theorem – every segment has exactly one midpoint – which they prove by dividing by two. But they never explain how a real number was assigned to the length or, after division, how to locate the midpoint. It just appears!

Construction 1.1 *Bisect an angle.* ***(Euclid, Book I, Prop. 9)***

Solution

Given $\angle EFG$, take any point J on $\overrightarrow{FE}$. There exists a point K on $\overrightarrow{FG}$ such that $\overline{FJ} = \overline{FK}$. Construct an isosceles triangle with **base** $\overline{JK}$ and **apex** L on the other side of $\overleftrightarrow{JK}$ from F. By SSS, $\overline{JFL} \cong \overline{KFL}$, which holds the equality $\angle JFL = \angle KFL$.

■

To construct an isosceles triangle when the base is given, a geometer sets his compass to any length longer than half the base and draws arcs from each endpoint. Where these arcs intersect is an apex; there are two possible, one on each side of the base. These arcs are each called a **locus**, and together, **loci** (lō’ sī). To construct an isosceles triangle when the apex angle is given, lay off the same arbitrary length on both rays from the **vertex** and then connect these points.

Construction 1.2 *Bisect a segment.* ***(Euclid, Book I, Prop. 10)***

Solution

Given $\overline{EF}$, construct an isosceles triangle with $\overline{EF}$ the base and G the apex angle. Using C. 1.1, bisect the apex angle, $\angle EGF$. (When finding G, swing your compass around to find G'' on the other side of $\overleftrightarrow{EF}$.) Let $\overrightarrow{GG''}$ cut $\overline{EF}$ at M. By SAS, $\overline{EGM} \cong \overline{FGM}$, which holds the equality $\overline{EM} = \overline{FM}$; that is, M is the midpoint of $\overline{EF}$, so $M \equiv M_{EF}$. ■

Moise (1990, p. 83) derides the "lighthearted use of the word *let*." Not us! We *proved* the crossbar theorem!

Construction 1.3 *Raise a perpendicular from a point on a line.* ***(Euclid, Book I, Prop. 11)***

Solution

Given a line with M on it, lay off the same arbitrary length to the left and to the right of M, so $\overline{EM} = \overline{FM}$. Construct an isosceles triangle with base $\overline{EF}$ and apex G. By SSS, $\overline{EMG} \cong \overline{FMG}$, which holds the equality $\angle EMG = \angle FMG$, so these are right angles.

■

Construction 1.4 *Drop a perpendicular from a point to a line.* ***(Euclid, Book I, Prop. 12)***

Solution

Given G not on $\overleftrightarrow{EF}$, construct an isosceles triangle with apex G and base $\overline{JK}$ on $\overleftrightarrow{EF}$. The apex angle bisector, $\overrightarrow{GG''}$ (construct it in the same way as in C. 1.2) cuts $\overline{JK}$ at M. By SAS, $\overline{JGM} \cong \overline{KGM}$, which holds the equality $\angle JMG = \angle KMG$, so these are right angles. ■

These constructions are the four basic techniques that will be used in combination throughout geometry. At the most fundamental level, all four are much alike. This is analogous to how the jab, hook, uppercut, and cross are the basic techniques that are used in combination throughout boxing. But all four involve giving somebody a poke in the nose, so they are much alike. Did you get the equilateral triangle theorem? You only had two theorems in your kit! Like a carpenter who only owns a claw hammer, for every nail, he is either going to hit it or pry it out. What else?

Construction 1.5 *Replicate an angle.* ***(Euclid, Book I, Prop. 23)***

Solution

Construct an isosceles triangle with the given angle as its apex angle by laying off equal lengths and connecting them. By SSS, reconstruct this triangle elsewhere.

■

Construction 1.6 *Given a ray and a point on the angle bisector, find the other ray of the angle.*

Solution

Given $\overrightarrow{EF}$ and P on the angle bisector, construct an isosceles triangle with apex E and base $\overline{PJ}$ with J on $\overrightarrow{EF}$ so $\overrightarrow{EJ} \equiv \overrightarrow{EF}$. By C. 1.5, construct $\angle KEP$ equal to $\angle JEP$ by using SSS to construct $\overline{KEP} \cong \overline{JEP}$ with J and K on opposite sides of $\overrightarrow{EP}$. $\overrightarrow{EP}$ bisects $\angle JEK$.

■

The perpendicular bisector of a segment is called its **mediator**. The perpendicular from a triangle vertex to the (extension of the) opposite side is the altitude. Altitudes and angle bisectors can be extended past the opposite side, but when lengths are assigned to an altitude or to an angle bisector, it means the length of the segment from the vertex to the opposite side.

Center Line Theorem

An angle bisector and a perpendicular bisector coincide if and only if the triangle is isosceles.

Proof

Assume the angle bisector and perpendicular bisector coincide. By SAS (segment reflexivity, the right-angle postulate and segment bisection), the two **right triangles** are congruent, so their hypotenuses are equal. Thus, the given triangle is isosceles.

•

Assume the triangle is isosceles. By the isosceles triangle theorem, the base angles are equal. Construct a median from the apex. By SAS (opposite sides, opposite angles, and bisection), the two triangles are congruent. The apex angle is bisected and the angles at the **foot** of the median are equal; both right because they bisect a straight angle.

• ■

The **center line** is the mediator of the base and the apex angle bisector of an isosceles triangle.

The center line theorem is **bi-conditional** and so it requires two independent proofs, concluded with •. The mediator theorem will also be like this. The two proofs may be done in either order.

Technically, p and q are equivalent even if proof that q implies p requires citing the previously proven statement that p implies q. However, students see it as a trick if I say, "prove that p and q are equivalent," but I do not mention that they must prove that p implies q first, and *then* prove that q implies p. No tricks! If this is the case, then I will call the statement that p implies q a theorem, and the statement that q implies p its converse, but I will not call them equivalent.

Interior and Exterior Angles Theorem

The bisectors of an interior and ***exterior*** *angle of a triangle are perpendicular to each other.*

Proof

Given $\overline{EFG}$ and J on $\overrightarrow{EF}$ past F, $\angle EFG$ is the interior angle and $\angle JFG$ is the exterior angle at vertex F. By C. 1.1, find K and L on the angle bisectors of $\angle EFG$ and $\angle JFG$, respectively. $\angle EFK = \angle GFK$ and $\angle JFL = \angle GFL$, so $\angle EFK + \angle JFL = \angle GFK + \angle GFL$ by addition. The union of these four angles is a straight angle and, if a straight angle is cut in two equal angles, then each one is right; thus, $\angle GFK + \angle GFL = \rho$ and $\overrightarrow{FK} \perp \overrightarrow{FL}$. ■

Mediator Theorem

A point is on the perpendicular bisector iff it is ***equidistant*** *from the endpoints of the segment.*

Proof

Assume that G is on the perpendicular bisector of $\overline{EF}$, but it is not M_{EF} (if it is, then we are done). By SAS, $\overline{EM_{EF}G} \cong \overline{FM_{EF}G}$, which holds the equality $\overline{GE} = \overline{GF}$. •

Assume $\overline{GE} = \overline{GF}$. Connect $\overline{GM_{EF}}$. By SSS, $\overline{EGM_{EF}} \cong \overline{FGM_{EF}}$, which holds the equality $\angle EGM_{EF} = \angle FGM_{EF}$. Thus, $\overrightarrow{GM_{EF}}$ is the angle bisector of $\angle EGF$ and, by the center line theorem, it is the perpendicular bisector of $\overline{EF}$. • ■

Problem 1.1 *Draw a line through a point so it cuts off equal segments from the rays of an angle.*

Solution

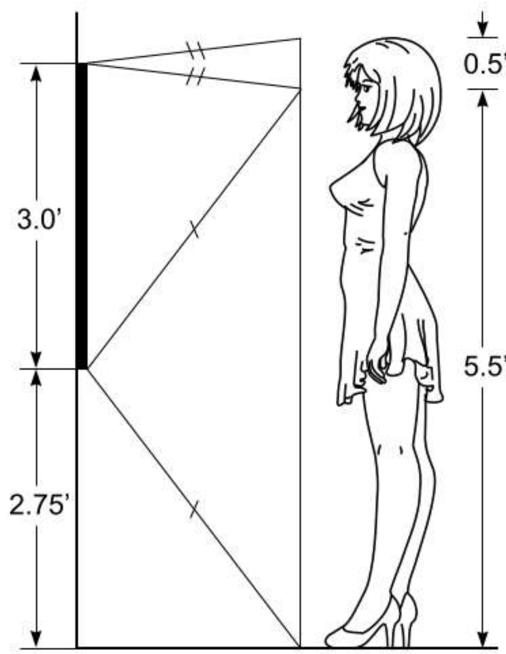
By the definition of isosceles, the desired line is the base of an isosceles triangle with the given angle at its apex. By the center line theorem, the base is perpendicular to the apex angle bisector. Bisect the angle and drop a perpendicular on it from the point. ■

Just solving a problem is not enough; you must also explain in what situations your solution might fail. One can always drop a perpendicular on a line, but not always on a ray, so this may not work. Sometimes there are two or more solutions to a problem, and you must explain why and under what conditions the number of solutions changes. This is called the **discussion**.

Problem 1.2 *A fink truss consists of an equilateral triangle built on the middle third of the ceiling joists. The rafters rest on the walls and meet at the triangle apex. Beams from the feet of the triangle meet the rafters at right angles. Draw it. The boards need not have width.*

Here, the roof's slope is $\frac{\varphi}{2}$; steep, but very strong. The king post and queen post trusses are more versatile, handling arbitrary and flatter slopes. Look them up if you are interested.

Problem 1.3 *Suppose your girlfriend asks you for a wall mirror. She is six feet tall in heels and her eyes are six inches below the top of her hair. What is the smallest mirror that allows her to see her entire self and how high should it be above the floor? Does it matter how far away she stands?*



Problem 1.3 Figure

Construct two isosceles triangles with bases from her eyes to her feet and to the top of her hair.

A carpenter constructs an A-frame with E and F the feet, G the apex, $\overline{EG} = \overline{FG}$ and a crosspiece between M_{GE} and M_{FG}, just like a commercial steel A-frame. But, when it is overloaded, the legs bow outward and start to pull free of the crosspiece. Reasoning that wood can take a compressive load but cannot pull things together while steel is just the opposite, he determines to connect $\overline{EM_{FG}}$ and $\overline{FM_{GE}}$ with wire rope to pull the bowed legs in tight with the crosspiece.

Problem 1.4 Suppose that you are the carpenter who built the A-frame described above.

1. There are two different ways to prove that $\overline{EM_{FG}} = \overline{FM_{GE}}$. Prove this both ways.
2. Another carpenter criticizes your design, stating that, if the wire ropes are both attached to an anchor hammered into the ground at M_{EF}, he can prove that the two wires are equal and, thus, that this is the best design. How do you respond?
3. You wish to build the strongest possible A-frame with the given boards and believe that this is accomplished by having the wire rope pull on the legs perpendicular to the bow in the boards. Prove that this is true if and only if $\overline{EFG}$ is equilateral.
4. Construct an A-frame with wire ropes from each foot to the trisection points of the opposite legs and with the bottom wire ropes meeting the opposite legs at right angles. Constructing this is too difficult; just draw it with a base of 13 cm and legs of 15.9 cm.

   Connect the upper trisection points with a board so you can tighten the wire ropes against it. If you agree to call this the "Aguilar A-Frame," I will make it easy for you by just telling you that the base is 81.65% of the legs' length. Note that, because the solution is a ratio, this must be a blue-belt problem. Geometry with multiplication!

White Belt Exit Exam

Saccheri Theorem I

*If $\overline{EFGH}$ is a **Saccheri quadrilateral**, so $\angle E = \angle F = \rho$ and $\overline{HE} = \overline{FG}$, then prove that*

1. $\overline{EG} = \overline{FH}$
2. $\angle G = \angle H$
3. $\overleftrightarrow{M_{EF}M_{GH}} \perp \overleftrightarrow{EF}$ *and* $\overleftrightarrow{M_{EF}M_{GH}} \perp \overleftrightarrow{GH}$
4. *The mediators of the base and the **summit** coincide.*

Rhombus Theorem

*Given a **rhombus** $\overline{EFGH}$, connect $\overline{FH}$. Without adding any **auxiliary** lines, prove that*

1. $\angle EFG = \angle GHE$
2. $\angle FGH = \angle HEF$
3. $\overline{FH}$ *bisects both* $\angle EFG$ *and* $\angle GHE$
4. *Draw the other diagonal,* $\overline{EG}$, *and prove that they are perpendicular bisectors.*

Isosceles Triangle Theorem Converse (White Belt)

If two angles of a triangle are equal, then the opposite sides are equal.

Perform these constructions:

1. *Construct a right triangle given one leg and the median from (a) that leg (b) the other leg.*
2. *Construct an isosceles right triangle so its apex altitude lies on a given line.*
3. *Construct an equilateral triangle so its apex altitude lies on a given line.*

Practice Problems

Construct each triangle using only the information given about it.

1.5 Construct a right triangle given the lengths of the legs.

1.6 Construct a triangle given the lengths of the three sides.

1.7 Construct a triangle given the apex angle and the lengths of the legs.

1.8 Construct a triangle given the lengths of the base, the median to the base and one leg.

1.9 Given $\overline{EFGH}$, if $\overline{EF} = \overline{GH}$ and $\overline{FG} = \overline{HE}$, prove that $\overline{EFG} \cong \overline{GHE}$ and $\overline{FGH} \cong \overline{HEF}$.

1.10 Given $\overline{EFG}$ with $\overline{GE} = \overline{GF}$, $\overrightarrow{GE}$ is extended to E'' and $\overrightarrow{GF}$ to F''. Prove $\angle FEE'' = \angle EFF''$.

1.11 Given $\overline{EFG}$ with $\overline{GE} = \overline{GF}$, construct an isosceles triangle, $\overline{EFJ}$, with the same base but not necessarily congruent to $\overline{EFG}$. Prove that $\angle GEJ = \angle GFJ$.

1.12 Given $\overline{EFG}$ with $\overline{GE} = \overline{GF}$, find points J and K on $\overline{EF}$ such that $\overline{EJ} = \overline{FK}$. Prove that J and K are also equidistant from the vertex; that is, $\overline{GJ} = \overline{GK}$.

1.13 The same as P. 1.12, but with J on $\overrightarrow{FE}$ past E, and K on $\overrightarrow{EF}$ past F.

1.14 Given $\overline{EFG}$ with $\overline{GE} = \overline{GF}$, prove that,

1. $\overline{M_{FG}M_{GE}M_{EF}}$ is isosceles.
2. $\angle GM_{GE}M_{EF} = \angle GM_{FG}M_{EF}$
3. $\angle EM_{EF}M_{GE} = \angle FM_{EF}M_{FG}$

1.15 Given two lines that intersect to make one right angle, prove that the others are also right.

1.16 Ancient hieroglyphics describe a $350'$ tall pyramid that had all the same dimensions as the Luxor hotel in Las Vegas, but it was reduced to rubble thousands of years ago. Could a *Common Core* student prove it congruent to the Luxor hotel by using superposition?

1.17 Your school has a foreign exchange student – from Mars! He accepts all our postulates except the parallel postulate. The symbol of his people is a $13 : 14 : 15$ triangle chiseled into a stone temple on Olympus Mons. He insists that it is not congruent to any Earthling triangle. By comparing rulers, you find that his unit of length is 3.219 cm. Can you draw a triangle and prove that it is congruent to his symbol? Can a *Common Core* student?

Pisa Tree Problem: You bought a laser rangefinder! *Yay!* But now, your geometry seon-saeng [teacher] has challenged you to measure the vertical height of the Pisa Tree in front of your school, so called because it leans like the Tower of Pisa. Because all the branches obscure your view, you cannot aim your laser straight up, so you take two measurements from either side: From an arbitrary point, you measure the distance to the treetop as 17 meters and, from 32 meters away and directly across from a point directly below the treetop (this is called its projection), you measure the distance to the treetop as 22 meters. What is the vertical height of the tree in meters? *Beware!* We have no assurance that our world is Euclidean and not hyperbolic, at any scale.

In the following constructions, you are not allowed to use a protractor. They are inaccurate when the angle is extended to the size of a house. Also, the students were not asked to buy one and so most of them did not. It is unfair for some students to use equipment the others do not have.

1.18 Construct an equilateral triangle, $\overline{EFG}$. In the *Notation* section, we define φ to be the interior angle of an equilateral triangle. Is this the same thing as defining it to be a third of a straight angle? Is $\overline{M_{EF}M_{FG}M_{GE}}$ equilateral in hyperbolic geometry, or only Euclidean? Can you prove that the interior angles of $\overline{M_{EF}M_{FG}M_{GE}}$ equal the interior angles of $\overline{EFG}$?

1.19 Given the hypotenuse, construct a half equilateral triangle. Is this a 30–60–90 triangle?

1.20 Inscribe a square in a given square. Now inscribe a different square in the given square.

1.21 Draw a king post roof truss with a right apex. The boards need not have width.

1.22 You wish to have black metal water pipes laid vertically on your roof, so the sun may heat water that is pumped through them. The plumbing supply store sells $45°$ elbows and you wish to use them so your

pipes bend over the apex of your roof and lay flat on both sides. Draw a king post roof truss with this apex angle. The boards need not have width.

1.23 The same as problem 1.22, but with $22.5°$ elbows. This is quite a flat roof, so it is like the king post roof truss, but with the addition of vertical boards from the rafter midpoints dropped onto the ceiling joists because the angled boards are at such a low angle that they do not fully support the midpoints of the rafters. The boards need not have width.

1.24 Draw a queen post roof truss with a right apex. In this design, lay off equal lengths from the apex onto the rafters, connect these points and drop perpendiculars onto the ceiling joist. It is not particularly strong for holding up a snow load, but it makes for a neat box shape in the attic that can be paneled as a room. The boards need not have width.

Comparison with *Common Core* Geometry*Common Core* teachers present the isosceles triangle theorem after showing students the button on *Geometer's Sketchpad* for bisecting a segment. They never demonstrate bisecting a segment with compass and straightedge; they rely heavily on that magical midpoint button.

Common Core Proof of the Isosceles Triangle Theorem (*Glencoe Geometry*, p. 286)

ΔLMP with $\overline{LM} \cong \overline{LP}$	Given
Let N be the midpoint of $\overline{MP}$.	Every segment has exactly one midpoint.
Draw an auxiliary segment $\overline{LN}$.	Two points determine a line.
$\overline{MN} \cong \overline{PN}$	Midpoint Theorem
$\overline{LN} \cong \overline{LN}$	Reflexive Property of Congruence
$\overline{LM} \cong \overline{LP}$	Given
$\Delta LMN \cong \Delta LPN$	SSS
$\angle M \cong \angle P$	CPCTC 😊

"Although a good proof of the theorem was known in antiquity, it has become customary in later centuries to prove it in needlessly complicated ways; and probably the worst of these rambling detours is the proof that starts by telling you to bisect [the apex angle] (Moise, 1990, p. 83)." C. 1.2 cites C. 1.1, so *Common Core* is this proof.

This is more complicated than the *Geometry–Do* proof: given $\overline{EFG}$ with $\overline{GE} = \overline{GF}$, $\overline{FGE} \cong \overline{EGF}$ by SAS, so $\angle EFG = \angle FEG$. It requires an auxiliary line (bisecting a segment is 4 more steps, so 12 total), but it is easier because students need not understand that the same three points can define different triangles depending on how they are ordered. This is important! Good job Grasshopper! You are still with us! Many got to page one and

wailed, "He just renamed $\overline{FGE}$ as $\overline{EGF}$. It's the same triangle!" Then they dropped out. When you are an engineer, one of them will vacuum your office.

The Common Core proof requires SSS and thus cannot be used in *Geometry–Do* because the proof of SSS requires the isosceles triangle theorem. David Coleman dodges the charge of circular reasoning by the simple expedient of not proving SSS. For him, SAS and SSS are both postulates or, if called theorems, they are "proven" with tracing paper. *Cheater! Cheater! Booger eater!*

Common Core states the triangle similarity theorem as an axiom – we prove it in the blue belt chapter – calling it either the similarity axiom or the dilation axiom, and then state without proof the AA, SAS and SSS similarity theorems. SAS, SSS, ASA, AAS and HL are then just special cases of the similarity/dilation axiom with the scale (dilation factor) being the multiplicative identity – which requires assuming the field axioms for real numbers – and the mid-segment theorem is a special case with the scale (dilation factor) being one half. *Common Core* students who claim to know of easier proofs to the isosceles triangle theorem and its converse can only say this because they did not have to prove SAS, SSS, ASA, AAS and HL. *Common Core* is just boring memorization!

The orange-belt chapter concludes with a section on how to pass a standardized exam of the type that is designed for *Common Core* students. Most of the people now reading these lines will not survive orange belt, so I will here tell you how a *Geometry–Do* white belt can pass *Common Core* exams. First, recognize that it is really an algebra exam in disguise, so review Algebra I. But the big secret is to bring a center-finding metric ruler and a compass to the exam so you can construct the figures – the ones provided are purposefully wrong – and measure the unknown quantity.

Varsity Tutors Advanced Geometry Exam, www.varsitytutors.com/advanced_geometry_diagnostic_1-problem-36916p, problem #14, is solved below, first using geometry, and then using the algebra that masquerades as geometry in *Common Core*. Which is easier?

Problem 1.25 *If a triangle has base 14 cm and legs 13 cm and 15 cm, what is its apex height?*

Geometry Solution

Use SSS to construct the triangle and then measure its height. It is 12 cm! ■

Algebra Solution

Let x and y be projections of the 13 cm and 15 cm legs onto the base, respectively. Then $x + y = 14$ cm and, by the Pythagorean theorem, $13^2 = x^2 + h^2$ and $15^2 = y^2 + h^2$. Solve both equations for h^2, set them equal and substitute $y = 14 - x$ into the latter.

$169 - x^2 = 225 - (14 - x)^2$

$$169 - x^2 = 225 - 196 + 28x - x^2$$

$$0 = -140 + 28x$$

$$x = \frac{140}{28} = 5 \text{ cm}$$

Substitute $x = 5$ into the first Pythagorean equation, $13^2 = x^2 + h^2$, then solve it for h.

$$h = \sqrt{13^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12 \text{ cm} \qquad \blacksquare$$

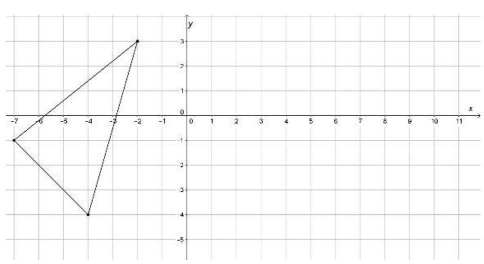
Varsity Tutors considers this advanced because almost no American geometry student can answer it correctly or, if they do, it takes them thirty minutes to work through all the algebra. But, if you construct the geometric figure with a ruler and compass (*Duh!* It is a geometry exam!), you can solve it in one minute using the most basic white-belt theorem you know.

Teachers! If you have read this far hoping for advice on how to get your #%$^@ students through the *Common Core* standardized exam, here it is: Ask for the perimeter of a triangle with vertices $(-2,3), (-4,-4), (-7,-1)$ and make it a race. The easy way is to lay the three sides end-to-end on a line. Taking the sum of three applications of the algebraic distance formula is the hard way.

$$\sqrt{(-2-(-7))^2 + (3-(-1))^2} + \sqrt{(-2-(-4))^2 + (3-(-4))^2} + \sqrt{(-7-(-4))^2 + (-1-(-4))^2} \approx 17.9$$

First-Day Exam in Geometry

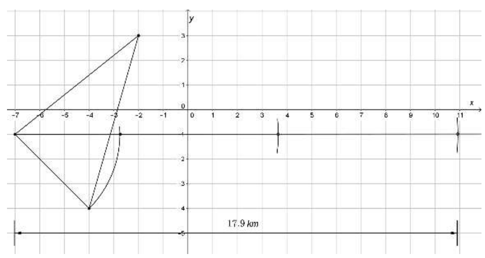
The first task of the high-school geometry teacher is to disabuse students of the notion that geometry is just a boring review of Algebra I. (Nothing new here. *Blah!!!*) You own a triangular pasture with vertices $(-2,3), (-4,-4), (-7,-1)$, as measured in kilometers. To the nearest 100 meters, how long is the fence around it? Make it a race with the *first* solver getting an A.



First Day Exam

The easy way is to lay the three sides end-to-end on a line. Put the compass pin at $(-7,-1)$ and rotate it to lay off the lower left side on the horizontal. Without moving the pin, measure the upper left side and lay it off on

the horizontal past the one you just did. Finally, measure the upper right side and lay it off on the horizontal past the one you just did. *It is segment addition!*



First Day Exam Solution

Taking the sum of three applications of the algebraic distance formula is the hard way to do this.

$$\sqrt{(-2-(-7))^2+(3-(-1))^2}+\sqrt{(-2-(-4))^2+(3-(-4))^2}+\sqrt{(-7-(-4))^2+(-1-(-4))^2}$$

$$=\sqrt{(5)^2+(4)^2}+\sqrt{(2)^2+(7)^2}+\sqrt{(-3)^2+(3)^2}$$

$$=\sqrt{25+16}+\sqrt{4+49}+\sqrt{9+9} = \sqrt{41}+\sqrt{53}+\sqrt{18} \approx 6.40+7.28+4.24 \approx 17.9 \text{ km}$$

The following are very easy geometry problems that will put *Common Core* graduates to shame:

Yellow Belt: Prove that the sum of the legs of a right triangle is less than twice the hypotenuse.

Orange Belt: Given a triangle with vertices $(0,0), \left(\frac{25}{3}, 0\right)$ and $(3,4)$, drop an altitude from the right vertex. What is the sum of the inradii of the three triangles thus formed?

White Belt Geometry for Construction Workers

Problem 1.26 *Rip a board into equal-width slats. (Three in this example.)*

Solution

Because no carpenter has ever made it to orange belt, I will here present the two transversals theorem unproven: Parallel lines that equally cut one **transversal** equally cut any transversal. Traverse the edges twice with a ruler held so it is easy to divide; *e.g.*, trisect a $5.5''$ wide board by angling the ruler so $0''$ and $6''$ lie on the edges. Set the saw guide so two thirds of the kerf is towards the edge and one third is towards the center.

■

George Birkhoff's axioms are called metric because they assume the field axioms for real numbers; those of David Hilbert and this author are called intrinsic because they do not. Birkhoff is assuming tape measures longer than one's workspace that do not droop and protractors that measure angles to such precision that they can be projected across one's workspace and the opposite side of the triangle is as accurate as can be measured with one's tape. Carpenters have no means of measuring angles with such precision and their tapes are only $25'$ long. The **Egyptian triangle** can verify that an angle is right, but it does not create a right angle. Finding the corners of a rectangle can be frustrating for carpenters who know only this. It works only if the sides are rigid and reach across the entire workspace, so there is no extrapolation error. The only time I recommend that construction workers use the Egyptian triangle is if they build an $8'$ wall, nail it to the floor, measure $6'$ from it, and then have two men stretch a tape diagonally; when their tape measures $10'$, nail the wall to the ceiling joists. It is vertical!

Squaring a $16'$ cabin is easy (the diagonal is $22'\ 7.5''$), but a rectangle with sides longer than a tape measure requires Thales' diameter theorem. No construction worker has ever made it that far, so I will break my vow against using unproven theorems and just present a cook-book recipe. A string can be extended six times longer than a tape measure and, because it is light weight, it does not droop when stretched across these long distances. Because a rectangle may be several times longer than your tape measure, you will need two strings in addition to your tape. Use a spring scale to put uniform tension on the string, about one Newton (100 grams) per meter.

Squaring a foundation must be achieved with no auxiliary lines outside it. This is because it may be in a hole if it is for a basement, or it may be surrounded by trees or cliffs if a plot of land was cleared and graded for a house being built in a forest or cut into a hillside. To make the house face a road, give the front the same compass heading as the center line of the road. To make the house face south, stand at the SW corner and aim $90°$ minus magnetic declination off magnetic north; *e.g.,* in Los Angeles, aim for $78°$ east. Note that this is a Euclidean construction.

Problem 1.27 *Square a house's foundation before pouring the concrete floor.*

Solution

Mark the front segment, $\overline{EF}$, with two stakes measured with a tape and oriented with a compass to be parallel to a road or to the east-west line; do not neglect declination. Loop the end of string S_1 over the E-stake, stretch it across the front and tie it to the F-stake. Drive a stake, O, into the ground near the center, but slightly towards the front and slightly towards the F-stake. Loop the end of string S_2 over the O-stake, stretch it to the F-stake, pinch it and then swing this radius around the O-stake until the arc intersects $\overline{EF}$. Drive in a stake at this intersection, E_1. Do not lose your pinched-off length! Lift the S_1 string off the E-stake and loop it over the E_1-stake. Stretch it over and past the center stake, O;

simultaneously, swing string S_2 around the O-stake to point in the opposite direction, away from E_1. Stretch both strings so they coincide (lie on top of each other) and drive a stake, G_1, in at the end of the length pinched-off on S_2. $\angle E_1FG_1$ is right by Thales' diameter theorem. Loop the end of string S_1 over the F-stake, stretch it into ray $\overrightarrow{FG_1}$ and drive a stake G on this ray past G_1 to where a tape measures the length of the side of the house. Pinch off this length, $\overline{FG}$, lift the S_1 string off the F-stake and loop it over the E-stake. Lift the S_2 string off the O-stake, loop it over the F-stake, pinch off the length $\overline{EF}$, then lift it off the F-stake and loop it over the G-stake. Stretch both strings out and where their pinched off lengths intersect, drive a stake, H. $\overline{EFGH}$ is a rectangle. Yay Thales! Down with Pythagoras! ■

This leads directly to a rectangle while the Pythagorean theorem converse (if $u^2 + v^2 = w^2$, then the triangle with these sides is right) is hit and miss. To "X it" is to measure the four sides to construct a parallelogram and then adjust it until the diagonals are equal. Like the Pythagorean theorem converse, it can verify a right angle, but its failure does not tell you how to adjust. P. 1.26 and P. 1.27 are orange- and green-belt, respectively, but we must help the carpenters, and they almost never get that far.

Many come to *Geometry–Do* with prejudice against deductive logic. Now is the time to rid ourselves of these losers! They are baggage we will not need to bring to yellow-belt geometry. Put construction workers and others who come to geometry with an open mind on Team Euclid. Put those who have closed their minds to deductive logic and believe only in coordinate geometry on Team Prástaro. In two classrooms, push the desks to the walls, staple butcher paper to the ceiling and draw a chalk line on it. Give each team a yardstick, two spools of chalked string and two ladders. A team that can draw a chalk line on the floor directly underneath the one on the ceiling gets an A, else an F. They cannot use a plumb bob, but you will test their answers with it. If the losers on Team Prástaro demand a tape measure instead of a yardstick, explain that, unless you are building an outdoor toilet, rulers are always less than the length of one's workspace.

The Egyptian or $3 : 4 : 5$ Right Triangle

In the preceding section I wrote, "Finding the corners of a rectangle can be frustrating for carpenters who know only this." So true! I remember when I was eight that my father had my mother, my brother and I at stakes marking three corners of the foundation of the basement for our house. He kept measuring sides one at a time with his only tape measure and ordering a stake moved a few inches this way or that. The Pythagorean equation never came out exact and it offered no hints on how to move the stakes to make it exact. Bad day!

In *Volume Two: Geometry with Multiplication*, the Pythagorean equation will be expressed as $u^2 + v^2 = w^2$ with u, v, w being real numbers. However, real numbers were only introduced in the 1800s and the modern

theory of rational numbers did not precede them by much. Yet Egyptologists assure us that triangles with sides of $3, 4$ and 5 units appear in four-thousand-year-old hieroglyphics. We will do the ancient proof and, in *Volume Two*, we will do it rigorously.

Egyptian Triangle Theorem

A triangle with sides three, four and five times a unit length is right.

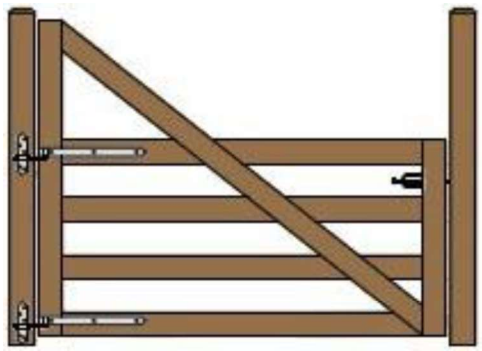
> ***Proof***
>
> Let F be on a line and G and G'' be on the line four units to each side of F. Let E be an intersection of circles of five-unit radii centered at G and G''. Observe that $|\overline{EF}| = 3$. By SSS, $\overline{EFG} \cong \overline{EFG''}$, which holds the equality $\angle EFG = \angle EFG''$. Thus, $\angle EFG = \rho$. ■

An analogous proof shows that a triangle of sides $5, 12$ and 13 is right; this was unknown to the Egyptians. Plumbers can use this triangle when installing $22.5°$ elbows. Integer solutions to the Pythagorean equation are known as Pythagorean triples. Students should be aware that Euclid devised a formula that generates Pythagorean triples: $u = m^2 - n^2,\ v = 2mn,\ w = m^2 + n^2$ for positive integers $m > n$. Verification is basic algebra; that ku, kv, kw for $k = 1, 2, \ldots$ gets them all is advanced. Try it with $n = 1$ and m even, or $n = 2$ and m odd.

$3 : 4 : 5$ right triangles are ubiquitous in *Common Core* because the programmers who compose their exams want to keep things neat by using only integers. *Varsity Tutors* Advanced Geometry Exam, problem #22 gives a rhombus of sides 5 units inscribed in a rectangle with height 4 units and asks the area. Problem 1.25 is the $3 : 4 : 5$ right triangle scaled up threefold and joined to the $5 : 12 : 13$ right triangle to be a $13 : 14 : 15$ triangle. A $13 : 20 : 21$ triangle has a 12-unit altitude for the same reason. A $15 : 20 : 25$ triangle is right – it is the $3 : 4 : 5$ right triangle scaled up fivefold; also, it is the threefold and fourfold $3 : 4 : 5$ right triangles joined along a 12-unit altitude; thus, it is the standard example of the geometric mean. Draw these triangles on your palm before exams and you have geometry mastered as *Common Core* defines the subject!

Basic Principles for Design of Wood and Steel Structures

As a geometer, you may be asked to design structures like gates, towers, gantries, or bridges. Everybody knows that a diagonal is required to make a rigid triangle, but a drive through the country indicates that few know which way it goes. Wood beams can withstand a tremendous compressive load – 1700 psi for Douglas Fir – but cannot lift a load because the screws pull out. Steel is just the opposite; $\frac{1}{8}$" wire rope can lift 340 pounds, but stainless-steel tubes kink and fold under any large compressive load. A single apostrophe means feet; a double apostrophe means inches.



A Badly Designed Gate

Wooden diagonals go from the foot of the gate post upwards and wire rope diagonals go from the top of the gate post downwards. For a wooden tower to be rigid, it must have crossed wooden diagonals so there are some that are angled upwards towards any direction of wind.

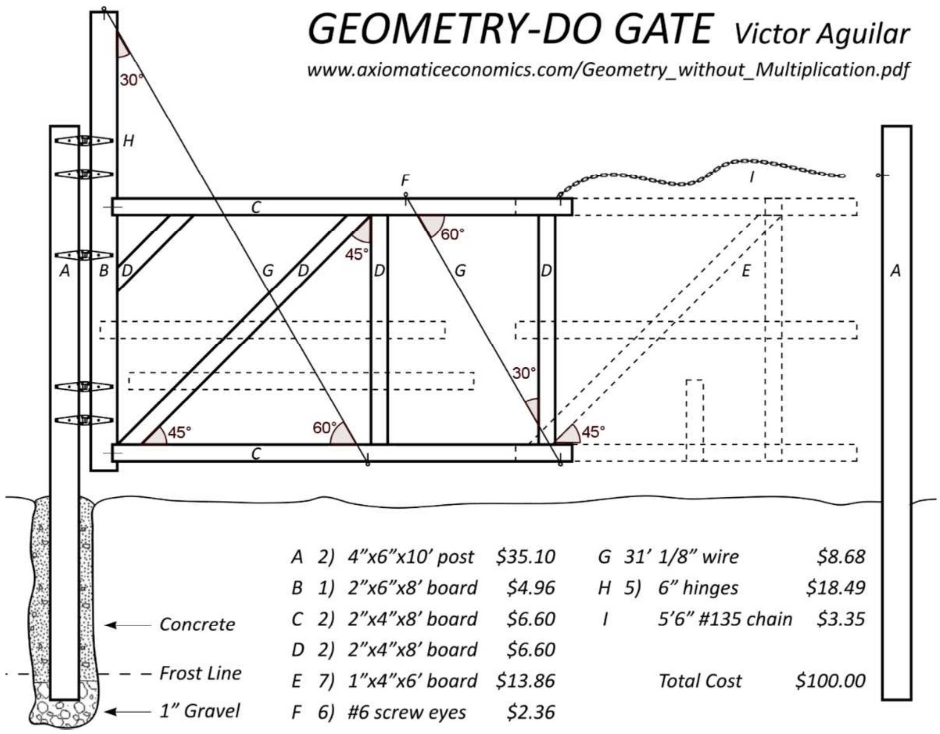
A gantry is two A-frames with a beam between them and a hoist that slides along the beam. Mimicking the all-steel commercial ones with wood does not work because, when overloaded, the legs spread apart and bow outwards. The crossbar is pulling them together, which is not what wood does well. Make the base of the A-frame 81.65% of the leg length and attach wire ropes from the feet to the trisection points; the lower one will meet the leg at a right angle. Install a wooden crossbeam between the upper trisection points to tighten the wire rope against.

A drive through the country indicates that almost all wooden gates have collapsed. This is because they have a wooden diagonal angled downwards and it reaches across the entire $12'$ or $14'$ gate, making too horizontal an angle. Also, failed gates were over-engineered on the latch side, adding unnecessary weight far from the hinges. $1''$ planks are all it takes to stop cattle.

The gate shown below is $14'$ wide for farm equipment and is designed to stop cattle, not people. The wire rope loops through the eyes and around both sides of the gate. Solid lines are $2''$-thick boards or $4''$-thick posts, dashed lines are $1''$-thick planks. Note that the boards and posts are all assembled edgewise, so their widest sides are coplanar.

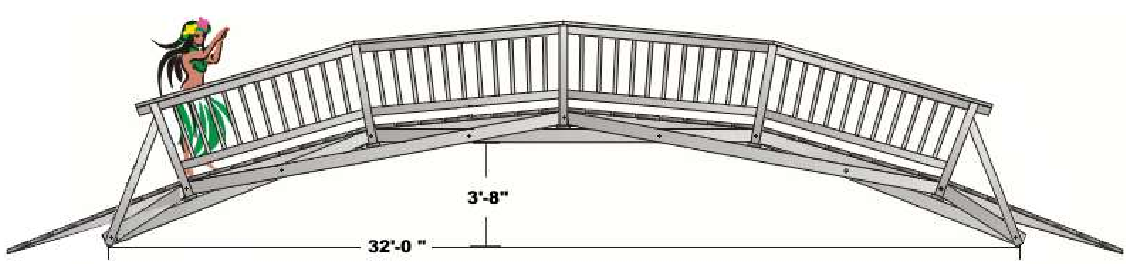
The C boards are inset into B and glued with wooden dowels to add strength. There are five hinges, and the gate post is rectangular; two hinges attached to a round post are weak. In the winter, the ground freezes to the frost line and, in the spring, the top few inches thaw but do not drain through the frozen ground below, which is why it is so muddy. Water that soaks into the gate post can only drain out the bottom if it extends below the

frost line. Also, there should be gravel, not concrete, below it to aid drainage. Gates often collapse because the post rots.



A Better Design for a Gate

For automotive bridges too high to be supported with pillars, put a $4'' \times 6'' \times 12'$ post vertical and two $4'' \times 4'' \times 4'$ posts at $45°$ under the center of each of the two stringers and lift them with $0.5''$ steel cables attached to eye bolts in the concrete footers. The two vertical posts should have crossed braces – it is a mistake to look only at the side view and neglect twisting forces. Yellow belts will learn to build stone bridges cut from river rocks that can support truck traffic!



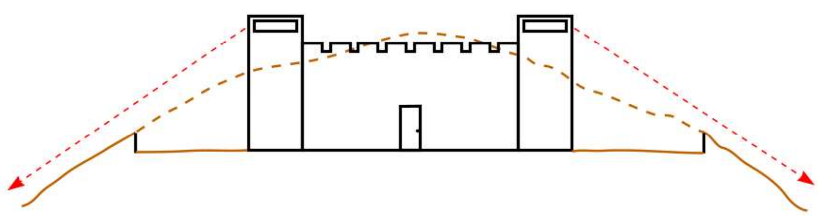
A Wooden Footbridge

Detailed plans for wooden foot bridges of various sizes are available. Sorry Grasshopper, but, while the plans are free, the hula girl coming out to dance on your completed bridge is extra.

Defense Positioning and Geometry

Steel cannons with rifled bores brought an end to state-sponsored castle construction, but some of what was learned during the time of smoothbore bronze cannons is still relevant today for people engaged in low-intensity conflicts. By low-intensity I mean, by mutual consent, both the homeowners and the bandits restrict themselves to small arms, usually defined as 7.62 mm rifles and hand grenades, because they are under a real army, but it will ignore small-arms fire.

The **first principle** of building fortifications is to build them for yourself, not for the enemy. If you dig trenches or set out some Jersey barriers, they may stop the enemy's wheeled vehicles, but they will also provide cover for enemy infantry. Thus, you should have a vertical retaining wall facing rearward and an earthen glacis slope facing forward. Flatten the top of a gently rounded hill, digging deep enough that the cut-down area requires a three- to four-foot-high retaining wall all around it. The windows are high enough that the defenders can graze the slope with rifle fire, but the attackers cannot fire at the base of the house wall until they crest the retaining wall.



An Example of a Glacis Slope

The **second principle** is to not have blind spots. The defenders should have bastions protruding from the corners of the building so attackers cannot press themselves up against the wall and be hidden from the windows. But, if the bastions are round, like the turrets in a medieval castle, there are blind spots directly in front of them. They should be tapered, like the points on a star.

An Example of a Star Fort

The **third principle** is that stone shatters when hit by bullets, but concrete and brick do not. Also, landscape with crushed stone to make walking noisy. Get rid of boulders that can be thrown.

There is little application for geometry in the design of fortifications, but white-belt geometers should be familiar with the basics. Green belts will learn of machine gun emplacement, which really does require geometry. It would be embarrassing for the *Geometry–Do* practitioner to boast of these advanced techniques while showing ignorance of basics like glacis slopes.

Next we turn to the positioning of troops along a frontier that is plagued with cross-border raids.

Having heartily mocked the NES for getting the equation for a parabola wrong (See the appendix. For a more thorough kicking of the *National Council of Teachers of Mathematics*, read my review of their geometry manual: www.researchgate.net/publication/335893456_Review_of_Essential_Understanding_of_Geometry), let us be more positive and look to the work of someone who knows what the terms directrix, focus and latus rectum mean. Raj Gupta (1993) wrote a book about the most basic function of an army: defense against cross-border incursions. It often happens that the politicians and high-ranking officers have the wit to understand that an all-out war would be disasterous for both countries. But small units will cross the border to pillage; they have the tacit approval of their officers, but they also know that, if they get in trouble, their officers will not send anyone to rescue them.

It is reasonable to assume that both the bandits and the defenders move with equal speed over the same terrain. Thus, the set of points equally distant from the defenders' base and from the border are where the enemy can be met on the run; points inside this graph are where the defenders can reach first, giving them a few minutes to lay machine guns and find depressions in the dirt for riflemen to lie in; points outside this graph are where the enemy can get past the base and must be intercepted by soldiers from another base. Here we are using the perpendicular length theorem, which states that the perpendicular is unique and is the shortest segment from a point to a line. This is proven by yellow belts; so, if you are white belt, please read ahead now.

For simplicity, we will assume that the border is locally straight. In Cartesian coordinates, we will make it the x-axis and label the base's coordinates $(0, 2w)$ with $0 < w$. (Having the parabola upward and with $h = 0$ is the simplest case. $w < 0$ requires using absolute value; switching x and y turns the parabola sideways. The orange-belt appendix on linear algebra explains how to rotate and, in that case, you need to know that the distance from a point, (x_0, y_0), to a line, $Ax + By + C = 0$, is $\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}$.) Consider the parabola, $y - k = \frac{1}{4w}(x - h)^2$. The point midway between the base and the border is on the desired graph and it is w distant from each of them. Make this point the vertex of the parabola so $h = 0$ and $k = w$. The parabola is $y - w = \frac{x^2}{4w}$ or $y = \frac{x^2+4w^2}{4w}$. The vertex is equally distant from the base and the border. If we can prove this

for every point on the parabola, then it defines the graph described in the previous paragraph. Thus, we must prove that the distance from (x, y) to $(0, 2w)$ is $y = \frac{x^2+4w^2}{4w}$, the distance from (x, y) to the x-axis. By the distance formula:

$$\sqrt{(x-0)^2+(y-2w)^2} = \sqrt{x^2+\left(\frac{x^2}{4w}-w\right)^2} = \sqrt{\frac{u^2}{16w^2}+\frac{u}{2}+w^2} \qquad \text{with} \qquad u = x^2$$

$$= \frac{1}{4w}\sqrt{u^2+8w^2u+16w^4} = \frac{1}{4w}\sqrt{(u+4w^2)^2} = \frac{x^2+4w^2}{4w} = y$$

Proven! When the bandits were teenagers, they were too busy thieving to have learned how to factor a quadratic. They will surely regret sleeping through Algebra I when you get in front of them with five minutes to spare! That is *plenty* of time to lay a squad automatic weapon and for your riflemen to find depressions in the dirt where they can settle into their shooting positions.

The discussion above defines the parabola for a single army base located $2w$ klicks from the border. (Klicks is an abbreviation for kilometers; clicks refer to increments of angle adjustment on an artillery piece.) But the more general question is, given the distance between army bases, how far should they be located from the border? The distance between the army bases is fixed by the budget; for instance, unless the politicians free up some more money, you may only have enough troops to man bases every four klicks. How far from the border should they be located to be sure that the bandits never get behind your line of bases? The parabolas of adjacent bases must intersect at or in front of the line of bases. So the question is, what must x be so that $y = 2w$? Since the parabolas of adjacent bases intersect halfway between them, the distance between bases is $2x$.

$$2w = \frac{x^2+4w^2}{4w} \quad \Rightarrow \quad 8w^2 = x^2+4w^2 \quad \Rightarrow \quad 4w^2 = x^2 \quad \Rightarrow \quad 2w = x$$

Thus, the distance between bases is $4w$, the width of each parabola when the bases are $2w$ from the border. In other words, the bases are twice as far from each other, $4w$, as they are from the border, $2w$. In the example above, if budget constraints require bases every four klicks, then the line of bases must be two klicks from the border. This is assuming that you know immediately when the border has been crossed – you are probably using electronic sensors – and your troops immediately move to intercept the bandits who are driving straight into your country. Of course, if Murphy has his way, none of these things will ever quite happen, so you will probably want to position your bases a little farther back – surrender a little more of your territory – to avoid allowing the bandits to ever get behind your line. Once into the interior, they are hard to catch.

Parabolas have other applications, so we must use more abstract language. The line that defines the border is called the **directrix**; the army base is at a point called the **focus**; and the segment parallel to the directrix that passes through the focus and has endpoints on the parabola is called the **latus rectum**. (No giggling allowed

when you hear the word rectum! It is Latin; it does not refer to any part of your anatomy.) For the parabola $y-k=\frac{1}{4w}(x-h)^2$, w is the distance from the vertex to either the focus or the directrix, $2w$ is the distance from the focus to the directrix, and $4w$ is the width of the latus rectum. For parabolas that were defined by the *National Evaluation Series* in which they got $\frac{1}{4w}$ upside down, the constant w is meaningless, as is most of what they teach. (The NES and many textbooks use c, not w, but that is confusing because c is the constant term in parabolas.) Screeching, *"It's just a constant!!!"* is not a valid argument. We need to end *Common Core*.

The preceeding two pages are algebra and, if taken as a review of a past Algebra I course, it can be taught at any time. If the geometry teacher will be absent and must ask the algebra teacher to substitute, the algebra teacher can lecture from these two pages. Next comes a Euclidean construction; purists will teach it in orange belt, though many will teach all four pages together.

To make this really useful to a field officer, we must teach the compass-and-straightedge construction of a parabola. An army captain does not have a computer in his tent with graphic-design software that allows him to superimpose a parabola on a map image. And he does not have a color printer to print out this new map. All he actually has in his tent is a desk and a paper map. The rule about the bases being twice as far from each other as they are from the border is something he can remember. But to make this really useful, he must draw the parabolas on his map so his mortar gunners can treat the outside of the parabolas as a free-fire zone while his ground troops can be careful to stay inside their parabola. Also, they may unroll concertina wire along the parabolas to slow the enemy, but leave gaps where – after the mortar gunners have ceased fire – the ground troops can exit to pursue enemy troops back to the border.

This construction is Euclidean because it assumes that parallel lines are everywhere equidistant.

1. With a colored pen, mark the location of the base and, if the border is not perfectly straight, draw a straight line that follows the border as closely as possible.

2. Drop a perpendicular from the base to the border and locate its midpoint. This is the parabola vertex; mark it another color. Measure this length in centimeters and call it w.

3. With a pencil, draw a line parallel to the border through the parabola vertex. Then draw a series of lines parallel to this line each one centimeter apart and continue past the base.

4. Set your compass to $w+1$ centimeters and, with pencil, draw arcs centered at the base that cut the first parallel line from the one that goes through the vertex. Mark these intersections with a dot of the same colored ink that was used for the vertex.

5. Repeat step #4 with a $w + 2$ cm arc intersecting the second parallel, then a $w + 3$ cm arc intersecting the third parallel, and so on for all the parallel lines.

6. Connect the dots with the same color of ink. You may want to free-hand this to make the parabola smoother than if it were composed of a series of straight segments.

7. Repeat this construction for each of the several bases in your area of operation.

The "twice as far from each other as they are from the border" rule depends on there being multiple units, like a four-platoon company. The captain decides where each platoon's base is to be built while the lieutenant in command of an individual platoon controls his troop's movements within their parabola and the shelling of targets outside their parabola. For instance, if the captain has four platoons and is tasked with guarding a straight length-M segment of the front line where $M = 16$ klicks, he will position bases 2 and 6 klicks inwards from the edges and 2 klicks back from the border. But what if he has only one unit to position in his area of operation?

Raj Gupta writes, "For an arbitrary probability density function of attack, all defending units must base themselves μ along the length-M front and distance σ inward from the border ($\sigma - \mu$ Theorem) in order to meet and defeat the invading forces as close to the front as possible (p. 8)." Proof of the $\sigma - \mu$ Theorem is beyond the scope of this book; but suffice it to say, μ and σ are the mean and standard deviation of the probability density function (pdf), respectively. (σ means a straight angle, but in statistics it is the symbol for standard deviation; on this page only, that is its use.) The pdf is an assignment of probabilities of the chance of attack at each point along the length-M front. The sum of the probabilities assigned to all the points on the front must add up to unity.

How are these probabilities assigned? This depends a lot on whether the enemy knows where your bases are. For the first line of defense, the platoon-size bases only two klicks from the border and featuring tall watch towers, it is obvious that they know. Thus, within each base's 4-klick wide subfront, there is zero probability of the enemy going hey diddle diddle, straight up the middle and a 50% probability of them attacking on either edge of the subfront. The mean is in the middle of the subfront, and the standard deviation is half the width of the subfront.

Thus, if each lieutenant were given the freedom of positioning his base anywhere behind his subfront regardless of what the other lieutenants are doing, logic would lead him and each of his fellow lieutenants to position their bases exactly as their captain would. Now suppose that the captain has a fifth platoon held in reserve. It does not have a watchtower and is positioned some distance from the front where it can move to reinforce any one of the four platoons. (Zulu chief Shaka had his reserves in a gully with their backs to the enemy and under orders

that, if any turned to peer over the edge, they would be shot. He did this to prevent them from attacking before they were ordered to.) Because the enemy does not know where it is, its probability density function is 20% at $0, 4, 8, 12$ and 16 klicks from one edge of its 16-klick front. Use your scientific calculator to find μ and σ, but do not use sample standard deviation like you would if you were doing confidence intervals. For this example, $\mu = 8$ klicks, the midpoint, and $\sigma \approx 5.657$ klicks back. Try this with different numbers of platoons. It is always optimal to have one reserve unit no matter how many bases there are; but the more bases, the closer the reserve can be to the front. If the enemy does not fear a platoon, they may invade anywhere; in this case, $\sigma = M/\sqrt{12} \approx 0.2887M$.

Glossary of White-Belt Terms

Altitude The perpendicular from a triangle vertex to the opposite side's extension

Angle Two rays, called the sides, sharing a common endpoint, called the vertex. $\angle F$ if there is one angle at F or $\angle EFG$ for the angle between $\overrightarrow{FE}$ and $\overrightarrow{FG}$.

Acute	An angle that is less than a right angle
Apex	The angle opposite the base of a triangle
Base	In a triangle with a base, the angles at either end
Complementary	Two angles that sum to one right angle
Exterior	The angle supplementary to an interior angle
Interior	An angle inside a triangle or quadrilateral at a vertex
Obtuse	An angle greater than right and less than straight
Right	The bisection of a straight angle
Straight	An angle whose rays are collinear and opposed
Supplementary	Two angles that sum to one straight angle
Vertical	Angles across from each other at an intersection

Apex The triangle vertex opposite the base

Arc Part of a circle; within equal circles, angles at the center and the arcs they cut off are a transformation of each other.

Area The measure of the size of a triangle or a union of disjoint triangles

Auxiliary Lines or arcs not given whose intersection goes beyond analytic

Axiom A proposition that is assumed without proof for the sake of studying the consequences that follow from it

Base The side of an isosceles triangle bracketed by the equal angles

The side of a triangle designated as such, or the one that it is built on

Between

1. If F is between E and G, then F is also between G and E and there exists a line containing the points E, F, G. (Between implies that the three points are distinct.)
2. If E and G are two points on a line, then there exists at least one point F lying between E and G and at least one point H such that G lies between E and H.
3. Of any three collinear points, there is exactly one between the other two.

Bi–Conditional A statement of the form p if and only if q. It is true if both p and q are true or both p and q are false. p implies q; also, q implies p. Proof of neither implication can cite the other implication. If and only if is abbreviated iff.

Bisect $\frac{1}{2}$ To divide a segment or an angle into two equal parts, called halves

Center Line The mediator of the base of an isosceles triangle or a semicircle

Circle All the points equidistant from a point, which is called the center

Collinear A set of points that are all on the same line

Congruent $\cong$ Two triangles whose areas and whose sides and whose interior angles are equal

Contradiction, Proof by To prove that statement p implies statement q, assume that p is true and q is not true and show that this is impossible.

Converse Given the statement that p implies q, the statement that q implies p

Convex Any segment between two points interior to two sides is inside the figure

Diagonal Segments connecting non-consecutive quadrilateral vertices

Definitional The adjacent side of the two triangles in a quadrilateral

Dichotomy Proof by contradiction when there are two alternatives

Discussion The necessary and sufficient conditions for a solution, and how many solutions

Disjoint Figures that do not overlap; their areas form an additive group

(This includes touching circles and adjacent triangles, if outside each other.)

Endpoint	A point at the end of a segment, arc, or ray	
Equal	Comparable magnitudes that are not less than nor greater than each other	
Equidistant	Two pairs of points that define two segments of equal length	
Equivalence	**Class**	A set of objects that are equal, congruent, similar, or parallel
	Relation	A set and a reflexive, symmetric and transitive relation
Equivalent	Conditions, any two of which are bi-conditional	
Extend	Given $\overline{EF}$, construct $\overline{EG}$ such that F is inside $\overline{EG}$ or E is inside $\overline{FG}$.	
Figure	A set of points either alone or joined in segments, rays, lines, arcs or circles	
Foot	The intersection when one drops a perpendicular from a point to a line	
Fully Defined	A figure with the given characteristics exists, and it is unique.	
Hypotenuse	The side of a right triangle opposite the right angle	
Inside	**Segment**	A member of the set of segment points, but not an endpoint
	Figure	A point such that any line through it intersects the figure at exactly two points and the point is between them
	Triangles	A triangle whose every point is inside of or on a side of another triangle, but the triangles do not coincide
Isometric	A transformation that preserves lengths; by SSS, it also preserves angles	
Legs	**Triangle**	The sides other than the base or the hypotenuse
Lemma	A theorem used for proving other more important theorems	
Length	The measure of the size of a segment; the distance between its endpoints	
Line	A segment extended past both endpoints; denoted $\overleftrightarrow{EF}$ if $\overline{EF}$ is the segment	
Locus	All the points that satisfy a condition; the plural is loci (lō’ sī)	

Magnitude	A set with both an equivalence relation, $=$, and a total ordering, $\leq$	
Median	A segment from a vertex of a triangle to the midpoint of the opposite side	
Mediator	The perpendicular bisector of a segment	
Midpoint	The point where a segment is bisected	
Neutral Geometry	A postulate set that does not mention parallel lines; absolute geometry	
Opposite	**In a Triangle**	An angle and a side across from each other
	Of a Line	Endpoints of a segment cut by the line
	In a Quadrilateral	Two sides or two angles across from each other
Ordering, Total	A set and a relation, $\leq$, that is not symmetric, but is reflexive, anti-symmetric and transitive	
Parallel	Two lines that do not intersect	
Perpendicular	A line whose intersection with another line makes a right angle	
Polygon	The union of multiple triangles adjacent on their sides such that it is convex	
Postulate	The axioms that are specific to geometry, not to other branches of math	
Quadrilateral	The union of two triangles adjacent on a side such that it is convex; $\overline{EFGH}$	
	Rhombus	A quadrilateral with all equal sides; plural, rhombi
	Saccheri	A quadrilateral with two opposite sides equal and perpendicular to the base
Radius	A segment from the center of a circle to the circle; plural, radii	
Ray	A segment extended in one direction; denoted $\overrightarrow{EF}$ if $\overline{EF}$ is the segment	
Reflexive Relation	A binary relationship over a set such that every element is related to itself	
Relation	A true/false operator on an ordered pair of elements from a given set	

Segment	All the points along the shortest path between two points; $\overline{EF}$	
Side	**Triangle**	One of the three segments that form a triangle
Summit	The side of a Saccheri quadrilateral that is opposite the base	
Symmetric Relation	A relation that can be stated of two things in either order	
Theorem	A statement requiring proof using postulates or already proven theorems	
Transitive Relation	If a relation is true for a and b and for b and c, then it is true for a and c	
Triangle	Segments connecting three noncollinear points, called vertices; *e.g.*, $\overline{EFG}$.	
	Acute	A triangle with all angles acute
	Degenerate	The vertices are collinear; this is not a triangle
	Egyptian	A triangle with sides 3, 4 and 5 units long
	Equilateral	A triangle with all sides equal
	Half Equilateral	An equilateral triangle cut at its center line
	Isosceles	A triangle with two sides equal
	Obtuse	A triangle with one angle obtuse
	Right	A triangle with one angle right
Trichotomy	Proof by contradiction when there are three alternatives	
Undefined Terms	Intuitive concepts: plane, point, shortest path, straight	
Under Defined	Not enough given information; the solutions are infinite in number	
Vertex	The intersection of two lines, rays, or sides of a triangle or quadrilateral	

Appendix: How Administrators Can Address Parents Concerned About *Common Core*

Everybody look and laugh at the *Common Core* equation for a parabola. Obviously, American high-school mathematics teachers are incapable of explaining how the Army defends against cross-border incursions as I do above in the section *Defense Positioning and Geometry*. Also, they cannot explain how to square a foundation.

NES Profile: Mathematics (304)

SECONDARY MATHEMATICS FORMULAS

Formula	Description
$V = \frac{1}{3}Bh$	Volume of a right cone and a pyramid
$V = Bh$	Volume of a cylinder and prism
$V = \frac{4}{3}\pi r^3$	Volume of a sphere
$A = 2\pi rh + 2\pi r^2$	Surface area of a cylinder
$A = 4\pi r^2$	Surface area of a sphere
$A = \pi r\sqrt{r^2 + h^2} = \pi rl$	Lateral surface area of a right cone
$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + a_n)$	Sum of an arithmetic series
$S_n = \frac{a(1-r^n)}{1-r}$ Wrong!!!	Sum of a finite geometric series This needs the condition $r \neq 1$
$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \ \lvert r \rvert < 1$	Sum of an infinite geometric series
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	Law of sines Is c the side of the triangle or $\frac{1}{4}$ of the latus rectum as shown below?
$c^2 = a^2 + b^2 - 2ab\cos C$	Law of cosines
$(x-h)^2 + (y-k)^2 = r^2$	Equation of a circle
$(y-k) = 4c(x-h)^2$ Wrong!!!	Equation of a parabola $(y-k) = \frac{1}{4c}(x-h)^2$
$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	Equation of an ellipse
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	Equation of a hyperbola

The *National Evaluation Series* (NES) are tests for prospective teachers in each high-school subject; 304 is mathematics. The fifteen formulas are all that are required to become a high-school math teacher and they do not even have to be memorized; the formula sheet can be taken into the exam. *The parabola equation is wrong!!!* When done right, c is the distance from the vertex to either the focus or the directrix; $4c$ is the length of the latus rectum. Also, it is stupid to use c (not w) for this *and* for the constant term in $y = ax^2 + bx + c$. It is no surprise to me that new teachers who scored in the top quartile on their collage entrance exams are nearly twice as likely to leave teaching than those with lower scores. They are appalled to discover that *Common Core* is teaching garbage!

Glencoe Geometry (p. 256) declares two triangles congruent, one with all its sides and angles labeled: $a = 38.4$ mm, $b = 54$ mm, $c = 32.1$ mm and $\alpha = 45°$, $\beta = 99°$, $\gamma = 36°$. The other triangle has the side corresponding to a labeled $(x + 2y)$ mm and the angle corresponding to β labeled $(8y - 5)°$. *Glencoe* then solves $8y - 5 = 99$ and $x + 2y = 38.4$ simultaneously to get $x = 12.4$ and $y = 13$. The former equation implies that y is an angle and the latter equation implies that x is something that, when added to an angle, is a length. *What?* x is the mysterious lengle!

Ridiculing *Glencoe Geometry* for adding a length to an angle is the best way to turn parents against *Common Core* because most of them do not remember the equation for a parabola from when they were in high school. But let us not overlook the fact that *Glencoe* set up a triangular system (one coefficient is zero) because this is all that is expected of *Common Core* students in 10th grade. When I was in high school – and this is still true in Europe – we solved two simultaneous linear equations with Cramer's Rule, and larger systems with Gaussian elimination.

Also worthy of ridicule is the practice of *Common Core* to attempt to give themselves the weight of antiquity. The Russian text of Kiselev's *Planimetry* is here: www.axiomaticeconomics.com/geometr-kiselev-1892.djvu Givental does not say he is doing an abridgment, but Kiselev has 302 sections and Givental has only 260. Also, Givental is a faithless translator. Равенство means equality, not congruence. Consider this: "**11. Равенство конечных прямых.** Два отрезка прямой считаются равными, если они при наложении совмещаются." This should be, "**11. Equality of finite lines.** Two line segments are considered equal if superposition makes them coincide." For Givental, this is section six because he is omitting sections. He translates it, "**6. Congruent and non-congruent segments.** *Two segments are congruent if they can be laid one onto the other so that their endpoints coincide.*"

Modern American textbooks use "congruent" to the complete exclusion of "equal," but this does not give the translator of an historical document the right to make this change unannounced. Also, notice that Givental talks around "superposition" because, while modern American textbooks rely on superposition, they attempt to write three grade levels below their students and hence omit this big five-syllable word in favor of "lay one onto the other." Givental is trying to make this 19th century Russian geometer sound like an early founder of *Common Core* geometry. He is not. While I do not agree with Kiselev on everything – specifically, I do not rely on superposition – *Planimetry* is a useful historical document. I cannot recommend this faithless translation.

Read the djvu file in Russian if you know the language. But, whatever you do, do not allow *Common Core* shills to convince you that their textbooks are based on the work of A. P. Kiselev, Felix Klein, or any other 19th century geometer. *Common Core* is based entirely on Bill Gates' desire to monopolize educational software sales. Gates (p. 3, usprogram.gatesfoundation.org/-/media/dataimport/resources/pdf/2016/12/geometry-outline2014.pdf) said "AAS is not sufficient for congruence," thus effectively banning *Geometry–Do* and every other geometry textbook that does not toe the *Common Core* line. It was at Gates' insistence that David Hilbert's good name be scrubbed, and Felix Klein be elevated to be the founder of all modern geometry. Klein was Hilbert's assistant, but the small original contribution he did make involved motion of geometric figures, which Gates latched onto because it helps him adapt his animation software to be sold as educational software.
So now geometry ceases to exist for students the moment they rise from their school computers running animation *cum* educational software sold by Gates for huge sums of taxpayer money. And no American carpenter can square a foundation without surveying equipment!

References

Birkhoff & Beatley. (2000) *Basic Geometry*. New York, NY: American Mathematical Society.

Carter, *et. al*. (2014) *Glencoe Geometry, Common Core Teacher Edition*. Columbus, OH: McGraw-Hill.

Euclid (2013) *The Elements*. Thomas Heath, trans. Santa Fe, NM: Green Lion Press.

Fraleigh, John B. (1989) *A First Course in Abstract Algebra*. Reading, MA: Addison-Wesley.

Gupta, Raj. (1993) *Defense Positioning and Geometry*. Washington, DC: Brookings Institution.

Hartshorne, Robin. (2000) *Geometry: Euclid and Beyond*. New York, NY: Springer-Verlag.

Hilbert, David. (1950) *The Foundations of Geometry*. La Salle, IL: The Open Court
math.berkeley.edu/~wodzicki/160/Hilbert.pdf
ia802805.us.archive.org/10/items/grunddergeovon00hilbrich/grunddergeovon00hilbrich_bw.pdf

Kiselev, Andrei Petrovich. (2006) *Geometry Book I, Planimetry*. Alexander Givental, translator. El Cerrito, CA: Sumizdat.

Mihalescu, Constantin. (2016) *The Geometry of Remarkable Elements*. Ioana Lazăr, translator. Titu Andreescu, *et. al.*, editors. Plano, TX: XYZ Press.

Moise, Edwin E. (1990) *Elementary Geometry from an Advanced Standpoint*. New York, NY: Addison-Wesley.

Wentworth & Smith. (2007) *Plane & Solid Geometry*. Online: Merchant Books.